Every (*non-constant, of course*) Polynomial Has a Zero
– Gauss and his fundamental theorem of algebra

Theory

1. The root formula of the quadratic equation $ax^2 + bx + c = 0$ is

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Or,

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

We also have $r_1 + r_2 = -\frac{b}{a}$ and $r_1 r_2 = \frac{c}{a}$.

2. If a number r is such that $r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$, r is called a zero of the polynomial $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n$, or a solution (a root) of the equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$. (The number n is called the degree of the polynomial.)

3. If r is a solution of the polynomial equation $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$, then there exist a unique positive integer m and a polynomial $q(x)$ such that $q(r) \neq 0$ and

$$x^n + a_1 x^{n-1} + \cdots + a_n = (x - r)^m q(x).$$

The number m is called the multiplicity of the solution r.

4. (Gauss' fundamental theorem of algebra) Every polynomial of degree n has exactly n (complex) zeros counting multiplicity:

$$x^n + a_1 x^{n-1} + \cdots + a_n = (x - r_1)^{m_1} \cdots (x - r_k)^{m_k},$$

$m_1 + m_2 + \cdots + m_k = n$.

Gauss: (*April 30, 1777- February 23, 1855*)

1. Which of the following statements is NOT true about Gauss' life?

A. His complete name was Johann Friedrich Carl Gauss.

B. Gauss' mother was well educated.

C. Gauss entered elementary school in 1784 and his teacher soon recognized his genius and devoted special attention to him.

D. His father had wanted him to follow in his footsteps and become a mason.

E. Gauss married twice and had 6 children. Two of his children immigrated to the United States. He did not want any of his sons to enter mathematics or science.

2. Which of the following statements is NOT true about Gauss' personality and interests?

A. He was a perfectionist and a hard worker.

B. He was deeply religious and conservative.

C. He loved philosophy and read Kant's *Critique of Pure Reason* five times.

D. He was rather depressed later in his life.

E. He published prolifically like Euler.

3. Which of the following statement is NOT true about Gauss' mathematics (I)?

A. The uniqueness of the factorization of integers into primes (or, the fundamental theorem of arithmetic) was first discovered by Gauss.

B. He introduced modular arithmetic into number theory.

$$123 = 2 \pmod{11}.$$

C. Gauss proved that the total number of primes less than n is approximately $\frac{n}{\ln n}$.

D. The famous "Bell Curve" describing probability distribution of certain random events was discovered by Gauss and is called now " the Gauss distribution".

E. He discovered a construction of the regular heptadecagon and using compass and straightedge.

4. Which of the following statements is NOT true about Gauss' mathematics (II)?

A. He introduced the notation i for $\sqrt{-1}$.

B. The least square method to minimize the impact of measurement error was invented by Gauss.

C. The elimination method of solving linear systems was first invented by Gauss and it is called now the Gaussian elimination method.

D. Gauss studied geometry of curves and surfaces. Gaussian curvature describes how a surface is curved.

E. Gauss' doctorate dissertation contains a proof of the fundamental theorem of algebra.

5. Which of the following statements is NOT true about Gauss' physics?

A. Gauss predicted correctly the position of the dwarf planet Ceres.

B. He was never a professor of mathematics but was appointed Professor of Astronomy and Director of the astronomical observatory in Göttingen.

C. He and Weber built successfully an electromagnetic telegraph.

D. He studied magnetism, crystallography, and optics.

E. He proved

$$E = mc^2.$$

Problems from AMC10

1.(AMC10A2003 5) Let d and e denote the solutions of $2x^2+3x-5=0$. What is the value of $(d-1)(e-1)$? (A) - $\frac{5}{2}$ (B) 0 (C) 3 (D) 5 (E) 6

2. (AMC10A2003 18) What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x+1+\frac{1}{x}=0?$$

(A) -$\frac{2004}{2003}$ (B) -1 (C) $\frac{2003}{2004}$ (D) 1 (E) $\frac{2004}{2003}$

3. (AMC10B2002 10) Suppose that a and b are nonzero real numbers, and that the equation $x^2+ax+b=0$ has solutions a and b. Then the pair (a,b) is (A) (-2,1) (B) (-1,2) (C) (1, -2) (D) (2,-1) (E) (4,4)

4. (AMC10A2002 16) If $a+1=b+2=c+3=d+4=a+b+c+d+5$, then $a+b+c+d$ is (A) -5 (B) -10/3 (C) -7/3 (D) 5/3 (E) 5

5. (AMC10B2005 16) The quadratic equation $x^2+mx+n=0$ has roots that are twice those of $x^2+px+m=0$, and none of m,n, and p is zero. What is the value of n/p? (A) 1 (B) 2, (C) 4 (D) 8 (E) 16

6. (AMC10B2003 24) The first four terms in an arithmetic sequence are $x+y, x-y, xy$, and x/y, in that order. What is the fifth term? (A) - $\frac{15}{8}$ (B)- $\frac{6}{5}$ (C)0 (D) $\frac{27}{20}$ (E)$\frac{123}{40}$

Problems from AMC12

1. (AMC12A2002 1) Compute the sum of all the roots of $(2x+3)(x-4)+(2x+3)(x-6)=0$. (A) 7/2 (B) 4 (C) 5 (D) 7 (E) 13

2. (AMC12A2002 12) Both roots of the quadratic equation $x^2-63x+k=0$ are prime numbers. The number of possible values of k is (A) 0 (B) 1 (C) 2 (D) 4 (E) more than four

3. (AMC12A2002 13) Two different positive numbers a and b each differ from their reciprocals by 1. What is $a+b$? (A) 1 (B) 2 (C) $\sqrt{5}$ (D) $\sqrt{6}$ (E) 3

4. (AMC12A2002 24) Find the number of ordered pairs of real numbers (a,b) such that $(a+bi)^{2002}=a-bi$. (A) 1001 (B) 1002 (C) 2001 (D) 2002 (E)2004

5. (AMC12A2001 19) The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the $y-$intercept of the graph of $y = P(x)$ is 2, what is b? (A) -11 (B) -10 (C) -9 (D) 1 (E) 5.

6. (AMC12A2001 23) A polynomial of degree four with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial? (A) $\frac{1+i\sqrt{11}}{2}$ (B)$\frac{1+i}{2}$ (C) $\frac{1+i2}{2}$ (D)$\frac{2+i}{2}$ (E)$\frac{1+i\sqrt{13}}{2}$

Problems from Putnam contest etc.

1. Prove that there are no prime numbers in the infinite sequence of integers

$$10001, 100010001, 1000100010001, \cdots .$$

2. Find a polynomial $P(x)$ such that $P(x)$ is divisible by $x^2 + 1$ and $P(x) + 1$ is divisible by $x^3 + x^2 + 1$.

3. Given the polynomial $F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integral coefficients $a_0, a_1, \cdots, a_{n-1}$, and given also that there exist four distinct integers a, b, c, d such that $F(a) = F(b) = F(c) = F(d) = 5$, show that there is no integer k such that $F(k) = 8$.

4. Prove that if $F(x)$ is a polynomial with integral coefficients and an integral zero, then for any positive integer k, at least one of $F(1), F(2), \cdots, F(k)$ is divisible by k.

5. Show that the greatest common divisor of $x^n + x - 1$ and $nx^{n-1} + 1$ is 1.

6. (2008 AIME) There exist unique positive integers x and y that satisfy the equation $x^2 + 84x + 2008 = y^2$. Find $x + y$.