Numbers, Numbers, Numbers
– Problems from Number Theory

1. Find the number of all divisors of the number $3! \cdot 5! \cdot 7!$.

2. For each positive integer $m > 1$, let $P(m)$ denote the greatest prime factor of m. For how many positive integers n, is it true that both $P(n) = \sqrt{n}$ and $P(n+48) = \sqrt{n+48}$? (AMC10A2005, 24) 1

3. For how many positive integers n does $1 + 2 + \cdots + n$ evenly divide $6n$? (AMC10A2005, 21) 5

4. How many positive cubes divides $3! \cdot 5! \cdot 7!$? (AMC10A2005,15) 6

5. For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1 = 2 + \cdots + n$? (AMC10B2005, 22) 16

6. Let $a_1, a_2, \cdots,$ be a sequence with the following properties.

(i) $a_1 = 1$, and (ii) $a_{2n} = n \cdot a_n$ for any positive integer n. What is the value of $a_{2^{100}}$? (AMC10?)

7. How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits? (AMC10B2003, 25) 30

8. Let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11? (AMC10A2003, 25) 8181

9. What is the units digit of 13^{2003}? (AMC10A2003, 16) 7

10. What is the probability that an integer in the set $\{1, 2, 3, \cdots, 100\}$ is divisible by 2 and not divisible by 3? (AMC10A2003, 15) 17/50

11. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, d, e, and $10d + e$, where d and e are single digits. What is the sum of the digits of n? (AMC10A2003,14) 12 (n=1533)

12. What is the largest integer that is a divisor of

$$(n+1)(n+3)(n+5)(n+7)(n+9)$$

for all positive even integers n? (AMC10B2003,18) 15

(More problems for high school students)

13. For how many positive integers n , $9+2^n$ is a perfect square? (1) Hint: $a^2-b^2=(a-b)(a+b)$.

14. For how many ordered pairs of positive integers (x,y), $\frac{xy}{x+y}=3^4$? (5) Hint: $xy-n(x+y)+n^2=(x-n)(y-n)$.

15. How many zeros does the number 1000! end with? (249)

16. What are the last two digits of 3^{1234}? (69)

17. Given positive integers n and m whose greatest common divisor, gcd(n,m), is 1, prove that there are integers s and t such that $sn+tm=1$.

Hint: induction.

18. If $gcd(n,m)=1$, prove $gcd(n-m,n+m)\leq 2$.

Remark: While some number theory problems are very hard to solve, the problems in this set are not too hard. Most problems are taken from past AMC10s and some are from past Putnam mathematical contests. Brief answers are given (except one) and hints are also given for some problems.