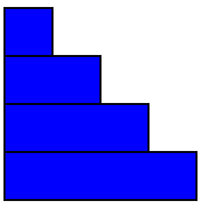
Sums and Geometric Proofs

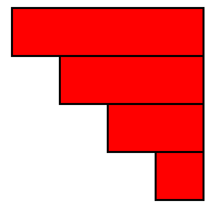
A series of formulas can be derived which tell us the sums of the first few powers of the integers. Three of these formulas are

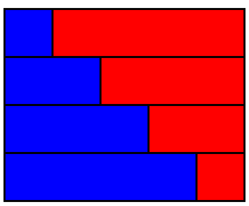
$$a_n = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2} n(n+1),$$
$$b_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6} n(n+1)(2n+1),$$
$$c_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4} n^2 (n+1)^2 .$$

A variety of methods can be used to prove these formulas. The question we wish to address here is if there is a geometric proof that can easily give us these formulas.



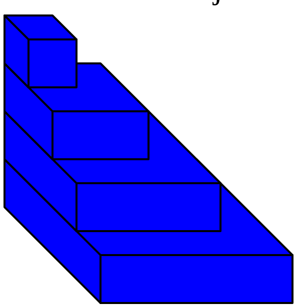
Consider first the sum of the first powers. Let us draw a geometric object in two dimensions whose area is the sum of the first n integers. For specificity, we will illustrate here and throughout with n = 4. Consider the blue object drawn at left, which consists of four rectangles joined together, each with a width of 1 and a length of i, with i running from 1 to 4. It is not immediately obvious how to find the area of this object.



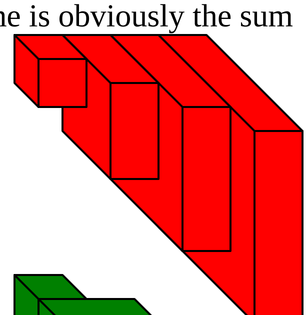


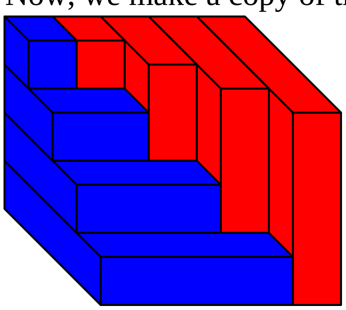
However, consider copying this object, and then rotating it until it looks like the red object at right. It now is an easy matter to combine these two objects to make a rectangle, as illustrated at left. The area of this object is 4 × 5 = 20, or generalizing to size n, it will have dimensions $n \times (n+1)$. Since we made this object using two regions of area a_n, we have

$$2a_n = n(n+1),$$
$$a_n = \tfrac{1}{2} n(n+1).$$



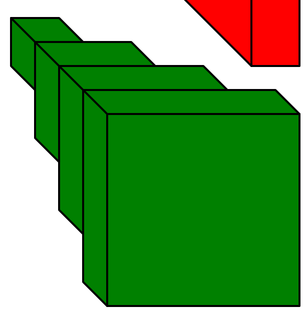
Can we generalize the success of this method to higher powers? Let's see if we can derive the formula for b_n. To do so, we must first draw an object whose volume is obviously the sum of the first n squares. Such an object is illustrated in blue at left. Each layer has dimensions $i \times i \times 1 = i^2$, so that the entire object has volume b_4. Now, we make a copy of this object, color it red, and rotate it as illustrated at right.

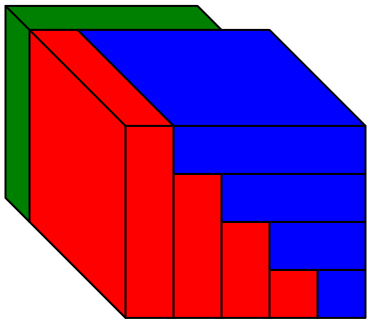




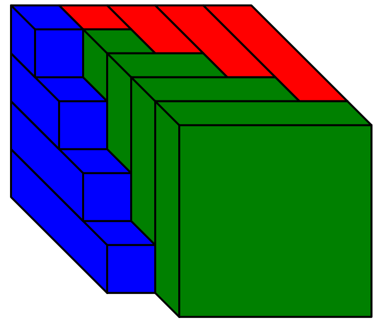
When you combine the red and blue shapes, you get a weird shape that looks no simpler than the original shape.

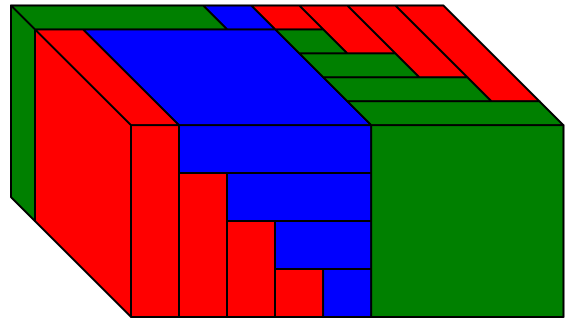
To make progress, we now make a third copy of the original shape, and then rotate it so that it looks like the green object seen at right. This fits snugly into the gaps left over from the red and blue structures, to make a nearly complete box, as shown below.





To complete the assembly, take three more copies of the structure, assemble them exactly as before, and then reflect the structure across all three planes, as illustrated at left. We now combine the two objects to form the box below right.





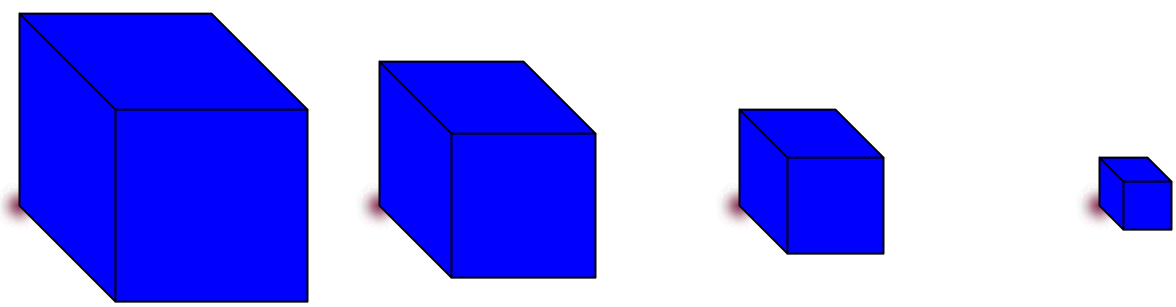
The final structure has dimensions 4 × 5 × 9, for $n = 4$. If we generalize to other values of n, it is clear that the same construction will work, and the dimensions will be $n \times (n+1) \times (2n+1)$. Since it was made using six objects, each of which had volume b_n, we have

$$6b_n = n(n+1)(2n+1),$$

$$b_n = \tfrac{1}{6} n(n+1)(2n+1).$$

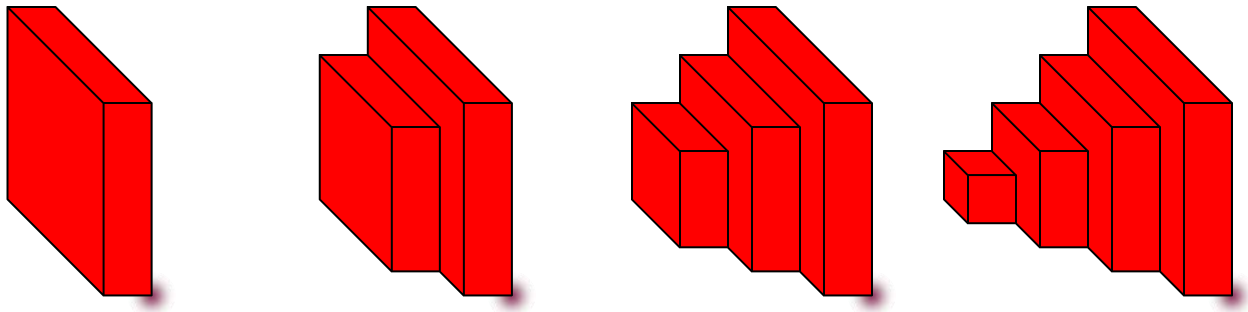
We are now in good shape to try to tackle c_n. There is just one very difficult problem: we will have to work in four dimensions. This problem is not insurmountable, but it makes the rotation of objects particularly difficult.

We start by trying to represent a four dimensional object that represents the sum of the first four cubes. To do so, we will slice the object into three dimensional layers, as illustrated below. This object is a depiction of a four dimensional object obtained by stacking four objects, each of dimensions $i \times i \times i \times 1$, on top of each other. To try to help us keep track of what is going on, I have marked one corner with a small purplish mark;

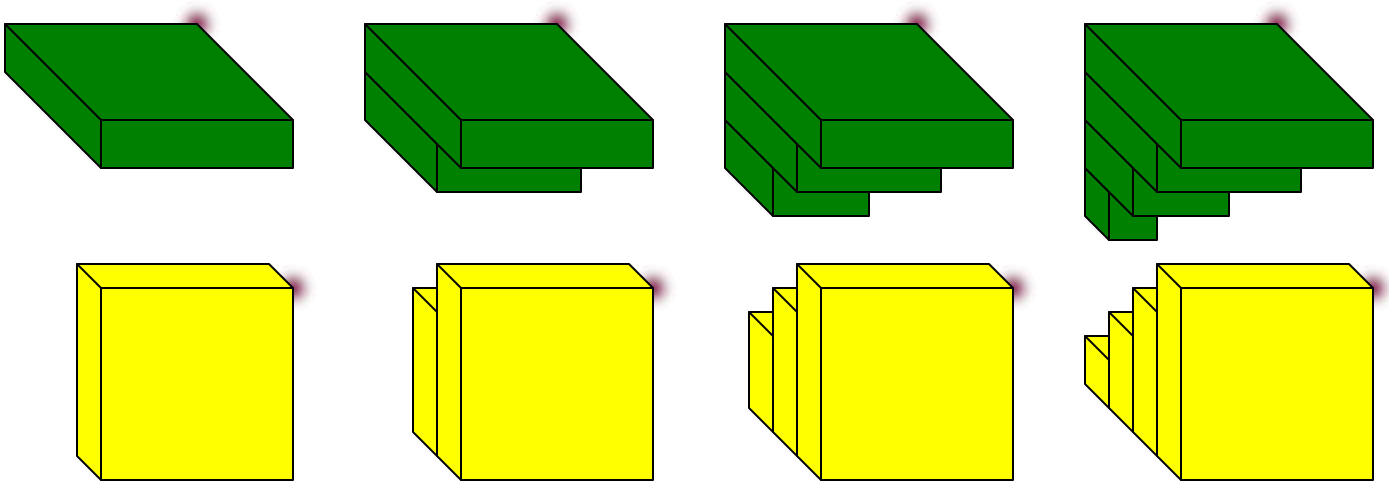


these points are directly "above" each other, separated only in the fourth dimension. In other words, each slab is stacked placed in the lower left back position "above" the cube "below" it.

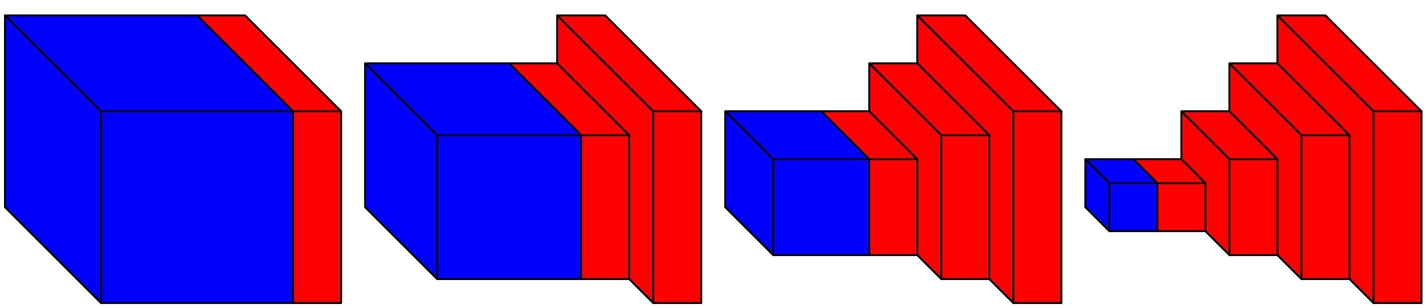
Now, based on our experience calculating c_n, it makes sense that we need to make more copies of this shape, rotated in various directions. This is where it gets difficult; it takes a great deal of thought to figure out what this object looks like when rotated in the fourth dimension. Each of the cubes will appear in multiple layers, the number of layers equaling the size of the cube. For example, one way to rotate this object looks like the following. The 4^3 slab appears in all four levels, while the 3^3 slab exists only on the top three levels, the 2^3 slab on the top two levels, and the 1^3 only on the top.



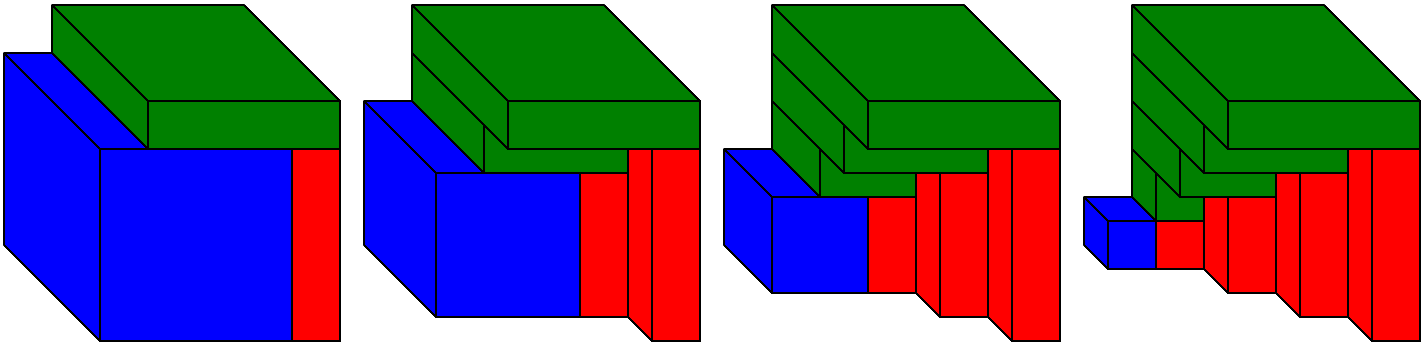
Two other ways of rotating this object are shown below.



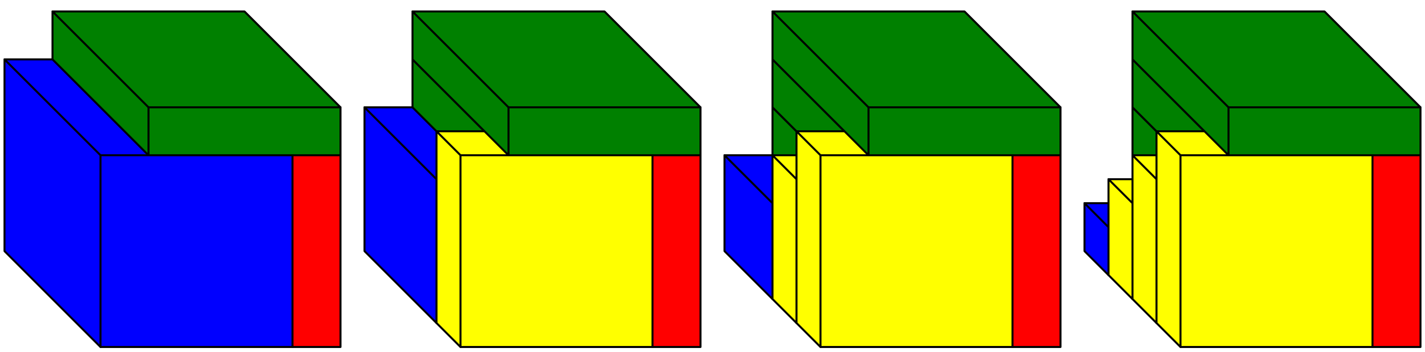
Now we would like to start assembling these structures into a larger object. First, we will put the red and blue structures together, to create the structure below.

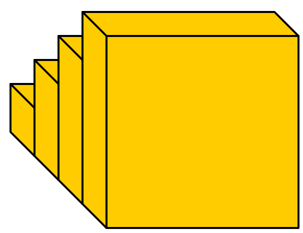


Now bring the green structure in from the top, to get the following:

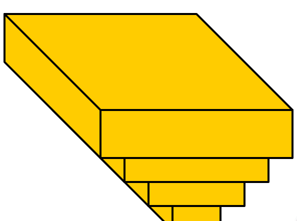


Now we start the trickiest part. We'd like to bring in the yellow structure, but this has to be done just right. We actually only want to bring in the bottom three levels of it, and in a non-trivial way. So we cut away level four, and then we shift the remaining structure up one level, so the lowest level gets inserted at level two, the second at level three, and the third level at level four, like this.

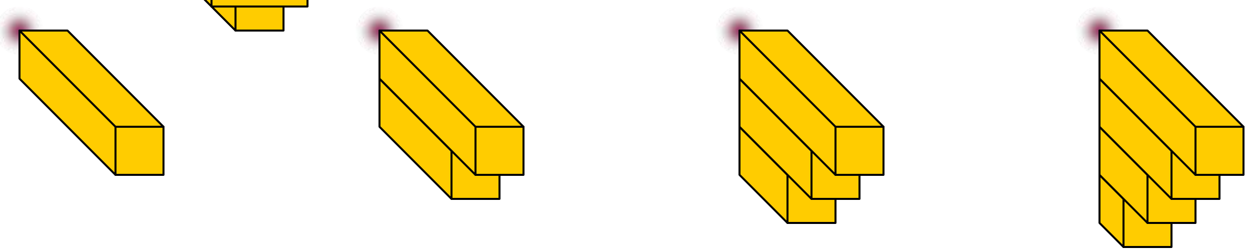




We now have nearly a four dimensional box, with just a little bit missing. We also have a leftover piece, which was removed from the yellow piece. It is drawn at right, but it has been colored orange to keep it distinct from the yellow piece from which it has been severed. Since it is all in the fourth "level," it has thickness one in the fourth dimension, with other dimensions as sketched.

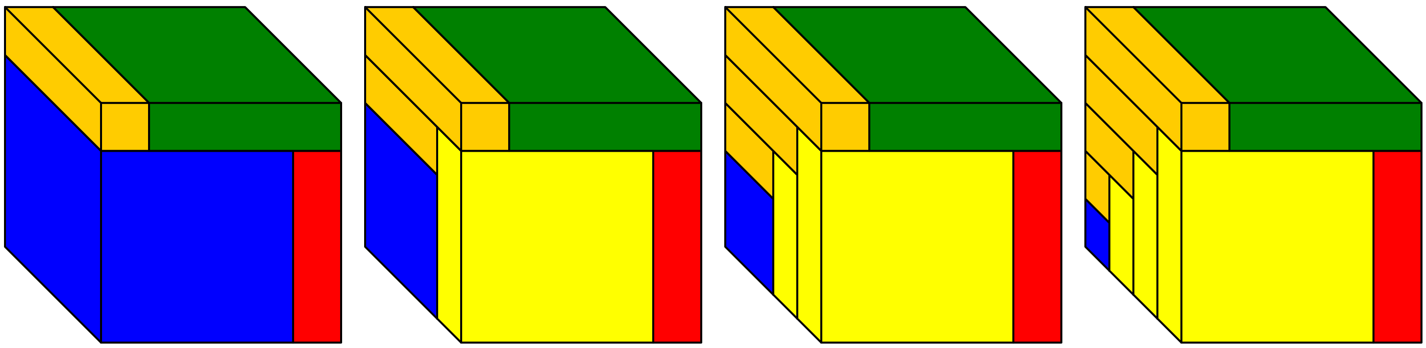


Now, we take this last remaining piece, and rotate it in four dimensions again. To try to illustrate this more clearly, I will proceed in two steps. First rotate it in the ordinary three dimensions, so that it looks like the object at left. Then take the horizontal (left to right dimension), and rotate the object into



the fourth dimension so that this dimension becomes the fourth dimension. The resulting shape is illustrated at the bottom of the previous page.

We now simply insert this structure into our previous shape, which yields a perfect four-dimensional box.



Keeping in mind that the largest blue cube on the left is size 4, this box is 4 × 5 × 5 × 4. There was nothing special about n = 4, in general, this would produce a four-dimensional box of hypervolume $n \times (n+1) \times (n+1) \times n$. Since we made this shape by assembling four objects, each of hypervolume c_n, it follows that

$$4c_n = n^2 (n+1)^2 ,$$

$$c_n = \tfrac{1}{4} n^2 (n+1)^2 .$$