Chapter 8

Rectangular Arrays

This is perhaps the most marvelous chapter among all nine chapters. I quote from the recent English translation and commentaries of the *Nine Chapters*: " This chapter makes a remarkable contribution to the development of mathematics." The Gaussian elimination method was clearly presented in this chapter in the general form. Gauss published his result in 1826, "about 2000 years later than the *Nine Chapters*". This chapter also introduces positive and negative numbers and rules of four basic operations.

The materials appeared in this chapter, solving systems of linear equations, will be studied in Algebra II and again in Linear Algebra when you go to college. However, in Algebra II, only systems of 3 unknowns are required. There are as many as 6 unknowns for some problems in this chapter.

The notation used in solving linear systems in this chapter coincides with the one we use today in Linear algebra: matrices. No variables are explicitly listed in the equations. Instead, the given data is arranged in the form of a *rectangular array* and operations are performed on the columns. We include a few problems in this set as an introduction to linear algebra. The Chinese language used to be written vertically. So, the given data is also displayed vertically, which is different from the convention in today's linear algebra textbooks.

Problem 1. Given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, and 1 bundle of low grade paddy, we gain 39 quarts of grain. 2

bundles of top grade paddy, 3 bundles of medium grade paddy, and 1 bundle of low grade paddy, yield 34 quarts of grain. 1 bundles of top grade paddy, 2 bundles of medium grade paddy, and 3 bundle of low grade paddy, yield 26 quarts of grain.

(Paddy: rice in the husk, growing, or gathered; rice in general; a rice field. In this problem, paddy means gathered ripe rice crop.)

Question: how much grain does one bundle of each grade paddy yield?

Solution: Top grade paddy yields $9\frac{1}{4}$ quarts per bundle. Medium grade paddy yields $4\frac{1}{4}$ quarts per bundle. Low grade paddy yields $2\frac{3}{4}$ quarts per bundle.

Method: List the given data as a rectangular array:

$$\begin{matrix} TopGrade \\ MediumGrade \\ LowGrade \\ quarts \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{bmatrix}$$

Perform the following operations on the columns to reduce the rectangular array to the desired form: the left upper triangle has only zero entries.

1. Middle $\times$ 3 - Right $\times$ 2. We have

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{bmatrix}$$

2. Left $\times$ 3 - Right.

$$\begin{bmatrix} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{bmatrix}$$

3. Left $\times$ 5 - Middle $\times$ 4.

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{bmatrix}$$

From the last rectangular array, we can easily get the answer for the low grade rice crop first. Then, use successive substitution to get answers for the medium and top grade rice crop.

In order for this method to work for general problems, it is necessary now to introduce the negative numbers and their operations. Ancient Chinese used counting rods of different colors to stand for positive (red) and negative (black) numbers. The color code is just the opposite of what we use today. Since I am not using colors in this document, I will use the negative sign for negative numbers.

Problem 2. Replace the data in Problem 1 with the one given below and find the answers to the same question.

$$\begin{matrix} TopGrade \\ MediumGrade \\ LowGrade \\ quarts \end{matrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now, try a problem with negative entries.

Problem 3. If we sell 2 cattle and 5 sheep to buy 13 pigs, we have a surplus 1000 coins. If we sell 3 cattle and 3 pigs, we can buy exactly 9 sheep. If we sell 6 sheep and 8 pigs, we still need 600 coins to buy 5 cattle. Find the price of each animal.

The data gives the following rectangular array.

$$\begin{matrix} Cattle \\ Sheep \\ Pig \\ coins \end{matrix} \begin{bmatrix} -5 & 3 & 2 \\ 6 & -9 & 5 \\ 8 & 3 & -13 \\ -600 & 0 & 1000 \end{bmatrix}$$

Problem 4. If you think these problems are fun to solve, here is another one. But I have removed the context of the problem.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & -1 \\ 4 & -1 & 0 \\ 1 & 1 & 11 \end{bmatrix}$$

If you like to set up the rectangular array yourself, try the following problem.

Problem 5. 1 county magistrate, 5 officials, and 10 servants can eat 10 chickens in total. 10 magistrates, 1 official, and 5 servants can eat 8 chickens. 5 magistrates, 10 officials, and 1 servant can eat 6 chickens.

Question: how many chickens can each of them eat?

There are also indeterminate systems in this chapter: the number of unknowns is bigger than the number of equations. There are also total 18 problems.

Chapter 9

Right-Angled Triangles

This chapter talks about the proofs and applications of the so-called "Pythagoras" theorem. It is called "Gougu" theorem in Chinese (even in today's textbooks). We talked about this theorem in the very first chapter. "Gou" and "Gu" refer to the two sides of a right-angled triangle.

Let c be the hypotenuse and a, b be the two sides. There are a couple of interesting identities that are equivalent to the "Gougu" theorem ($c^2 = a^2 + b^2$). Try to use your algebra skills to show that they are true identities.

$$2(c+a)(c+b) = (a+b+c)^2.$$

$$2(c+a)(c-b) = (a-b+c)^2.$$

For our high schoolers, try to solve the following problems.

Let c be the hypotenuse and a, b be the two sides.

1. Show that a must be a root of the cubic equation

$$x^3 + \frac{c-a}{2}x^2 = \frac{a^2b^2}{2(c-a)}.$$

2. Show that b must be a root of the biquadratic equation

$$x^4 + a^2x^2 = c^2b^2$$

Problem 1 Given a large circular log with a diameter 2.5 meters, assume it is turned into a rectangular plank of 0.7 meters thick. What is the (maximal) width?

Problem 2 Given a tree 20 meters high and 1 meter in circumference, a vine winds around it (evenly) 7 times from its root to its top. Find the length of the vine.

Problem 3 There is a reed in a pond. Its top is 1 meter above the water and is 5 meters from the bank. When it is drawn to the bank, its tip can barely touch the bank. Find the depth of the water and the length of the reed.

Problem 4 There is a rope hanging from the top of a pole. The extra length lying on the ground is 3 meters long. When tightly stretched, it is 8 meters from the foot of the pole. Find the length of the rope.

Problem 5 There is a wall 10 meters high. A pole leans against the wall so that its top is even with the top of the wall. If the foot of the pole is moved 1 meter further from the wall, the pole will lie flat on the ground. Find the length of the pole.

Problem 6 There is a circular log of unknown size (buried in a wall). When sawn (along the length) 0.1 meter deep, it shows a breadth of 1 meter. Find the diameter of the log.

Problem 7 There is a gate. When partially opened, it is 1 meter away from the threshold and the the gap between the halves is 0.2 meter. Find the width of the gate.

Problem 8 There is a door whose diagonal is 4 feet longer than the width and 2 feet longer than its height. Find the size of the door.

Problem 9 There was a bamboo 10 meters high. It was broken during a storm and now its tip is on the ground 3 meters away from its root (it is connected at the breaking point). Find the height of the breaking point.

Problem 10 There is a right-angled triangle with two sides 5 and 12. Find the side of the inscribed square.

Problem 11 There is a right-angled triangle with two sides 8 and 15. Find the diameter of the inscribed circle.

Problem 12** Find the side of the inscribed square of a general triangle with sides a,b,c.

Problem 13** Find the diameter of the inscribed circle of a general triangle with sides a,b,c.

Problem 14 A castle is surrounded by tall walls forming a square of sides 200 meters each. Four gates are located in the middle of each wall. At 15 meters outside the east gate, there is a tree. How far do people need to walk south from the south gate in order to see the tree (assuming there is no obstacles other than the wall.)?

Problem 15 There is a square city of unknown side, with gates opening in the middle of each side. 20 meters north of the north gate there is a tree, which is visible when one goes 14 meters outside the south gate and then 1775 meters westward. Find the length of each side (of the city).

Problem 16 (Paraphrased from Sharon's piano teacher Anne Listokin) There is a deep moat around a square city. The moat is 15 feet wide. A person has only two 14 feet long planks of wood (strong enough for him to walk on) and nothing else. Can he walk cross the moat using these two planks? (assume the land outside the city is flat, and the four corners of the moat are right-angled.)