Chapter 5

Engineering Consultation

The rule for a City Wall, Wall, Dyke, Trench, Moat and Canal is the same:

Add the upper and lower breadths, then halve; multiply by the altitude or the depth; then multiply by the length, giving the volume – Nine Chapters

In modern mathematics, finding the volume of a solid starts with the definition of the volume of a rectangular solid:

$$V = length \times width \times height.$$

Some simple solids' volumes can be obtained by using the method of trimming and patching. The rule given above can be justified easily with this method.

Problem 1 Now given a city wall with a lower breath of 12 meters and upper breath of 6 meters, an altitude of 15 meters and a length of 500 meters. Tell: what is the volume.

Unfortunately, most common solids' volumes can not be obtained with this trimming and patching method. These solids include the ball, the cone, the tetrahedron, the cylinder, etc.

Whether or not the volume of a regular tetrahedron can be obtained by using trimming and patching was one of the 23 problems Hilbert raised for the 20th century in 1900. The answer is no.

In this set of problems, we are allowed to use the volume formula for a pyramid of any shape: a cone, a tetrahedron, etc.

$$V = \frac{1}{3} height \times Base\ Area.$$

Problem 2 Find the volume of an icecream cone whose base perimeter is 24 cm and the height is 15 cm.

While it is easy to use the formula to find the volume of a cone, it is not so easy to find the volume of a frustum of a cone: a cone with the top part being cut off so that the circular top and the bottom are parallel.

Problem 3 Assume that we measure the top and bottom circumferences and we have 12 cm and 24 cm, respectively. The height is 10 cm. What is the volume of the frustum?

Problem 4* Find the formula for the volume of the frustum of a cone with the top and bottom circumferences C_1 cm and C_2 cm, respectively. The height is h cm.

We may think these two problems are easy. Let us try to challenge ourselves with some harder problems by cutting off the tip of the cone at an angle and then try to find the volume of the solid we have obtained. We may need the area formula of an ellipse: πab, where a, b are half of the long axis and the short axis, respectively.

Problem 5*** Assume that we have a cone with a base circumference C and a height h. Cut the tip of the cone off with at an angle so that the top is an ellipse. The highest point and the lowest point of the ellipse are h_1 and h_2. Find the volume of the rest of the cone.

Problem 6 Find the volume of a square pyramid whose base length is 100 meters and the height is 150 meters.

Problem 7 The tip of the pyramid in Problem 6 is again cut off. It is now a frustum with a height 120 meters. Find its volume.

Problem 8 Someone found a pyramid in a desert. But the top of the pyramid was gone. So it was really a frustum. He measured that the length of the bottom square was 100 meters and the length of the top square was

20 meters. The height of the frustum was 50 meters. What is the volume of this frustum?

Problem 9 A tetrahedron is a pyramid with a triangular base. It has total 4 triangular faces. Assume now we have a regular tetrahedron with the side length all equal to 10 cm. What is its volume?

Problem 10* We have a regular tetrahedron with the side length all equal to 10 cm. If we cut the top off at the 2/3 of the height. That is, the leftover frustum has a height equal to 2/3 of its original height. What is the volume of the frustum?

Problem 11** We have a beam that looks like prism – but two ends are trapezoids of different sizes. At one end, the upper width is 2 inches and the bottom width is 4 inches and the height is 4 inches. At the other end, the upper width is 3 inches and the bottom width is 6 inches and the height is 5 inches. Two trapezoids are parallel to each other and perpendicular to the beam in some sense. The length of the beam is 8 feet. Find its volume.

Problem 12* We now justify the volume formula of a cone. Assume the height is 9 and the radius of the bottom is 5.

We need to cut the cone horizontally into many thin disks. If we cut the cone into n disks – we have actually $n-1$ frustums and a tiny cone on the top. But anyway, we call all of them disks because we will treat them as if they were disks.

(1) What is the thickness (the height) of each disk?

(2) If we count from the top to the bottom, what are the radii of the first, the second, the third, and the fourth disk's bottoms, respectively?

(3) Treat each disk as if it were a cylinder with the radius of the base equal to the radius of the bottom of the disk. Find the approximate volumes of these disks.

(4) Add up the approximate volumes of these n disks and pull out the common factors including π. Do you see a sum of squares of whole numbers.

(5) Use the formula we had in Problem Set 2 to find the sum in (4).

(6) Find the destination of the number you get in (5) as n gets larger and larger – a calculator is needed now.

(7) The answer should be $75\pi = \frac{\pi r^2 h}{3}$.