Chapter 4

Shortening the Width

For convenience, the units have been changed.

Problem 1. 1 acre = 4840 square yards.

(1) Given a rectangular field whose width is $1\frac{1}{2}$ yards. Assume the area is 1 acre. What is its length?

(2) Given a rectangular field whose width is $1 + \frac{1}{2} + \frac{1}{3}$ yards. Assume the area is 1 acre. What is its length?

Problem 2. Find the first significant (non-zero) digit of the following square roots. Do not use a calculator.

(1) $\sqrt{1.919}$, $\sqrt{3.919}$, $\sqrt{4.0919}$, $\sqrt{13.919}$, $\sqrt{23.919}$, $\sqrt{73.0919}$, $\sqrt{99.919}$.

(2) $\sqrt{0.1919}$, $\sqrt{1.919}$, $\sqrt{19.19}$, $\sqrt{191.9}$, $\sqrt{1919.00}$, $\sqrt{19190}$, $\sqrt{191900}$.

Problem 3. Find the first significant (non-zero) digit of the following cubic roots. Do not use a calculator.

$$\sqrt[3]{1.919},\ \sqrt[3]{4.0919},\ \sqrt[3]{13.919},\ \sqrt[3]{239.19},\ \sqrt[3]{730.919},\ \sqrt[3]{999.19}.$$

4.1 Finding an approximate value of a square root

There are numerous ways to find an approximate value of a square root. In the book *Nine Chapters*, the following method is used.

4.1.1 Algorithm 1: Ancient Chinese Way

Given a number x. Assume $1 \leq x < 100$. If not, either divide or multiply x by $100, 10000$ etc. to bring the number into the interval. After finding the approximate value of the square root of the new number, divide or multiply it by $10, 100$ etc.

Step 1 Find the first digit of the square root and record it as a.

Step 2 Subtract from x the square of a, a^2, the result is called $y = x - a^2$.

Step 3 Multiply y by 100. Find another digit b so that

$$(20a + b)b \leq 100y < (20a + b + 1)(b + 1).$$

Subtract $(20a + b)b$ from $100y$.

Step 4 If the result from Step 3 is 0, terminate the process. The square root is $a + b/10 = a.b$.

If the result is not 0, it is called a new $\widetilde{y}$ and repeat Step 3 with a new $\tilde{a} = 10a + b$.

The approximate value of the square root is $a.bcd \cdots$ Continue until either the process terminates or the desired precision is reached.

Problem 4. Find the square root of 3 using Algorithm 1 up to 3 decimal places.

Problem 5. Find the square roots of

$$55225, 25281, 71824, 564752.25, 3972150625$$

using Algorithm 1. (These numbers are from the book *Nine Chapters*.)

Problem 6*. Use algebra to justify Algorithm 1, or tell a friend why the method works.

4.1.2 Algorithm 2: Shortening the Width

Assume $1 \leq x < 100$.

Step 1 Find the first digit of the square root, a. Let $b = x/a$.

Step 2 If $a = b$, terminate the process. If not, then $b > a$, go to Step 3.

Step 3 Calculate $\tilde{b} = b - (b - a)/2 = (a + b)/2$. Calculate $\tilde{a} = \frac{x}{\tilde{b}} = \frac{2x}{a+b}$.

Use $\tilde{a}$ and $\tilde{b}$ as new a, b and go to Step 2.

Repeat the process, the value a approximates the square root of x.

Problem 7. Use Algorithm 2 to find the square root of 3 accurate up to three decimal places.

Problem 8* Justify Algorithm 2 with algebra.

4.1.3 Algorithm 3: Continued Fraction

Assume that we want to find the square root of a number $x : 1 \leq x < 100$.

Step 1 Let a_1 be the first digit of the square root of x, $b = x - 1$.

Step 2 Calculate $a_2 = \frac{b}{2+a_1}$.

Step 3 Repeat Step 2: $a_n = \frac{b}{2+a_{n-1}}$

The sequence $\{a_1 + 1, a_2 + 1, a_3 + 1, \cdots, a_n + 1, \cdots\}$ approaches $\sqrt{x}$. We may say, the *destination* of the sequence is $\sqrt{x}$.

Problem 9 Calculate

$$1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+1}}}}}}.$$

Problem 10* Explain why this continued fraction approaches the square root of $a+1$.

$$1+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cdots}}}}}}.$$

4.1.4 Algorithm 4: Newton's Method, or the Ancient Egyptian Way

Assume that we want to find the square root of a number $x : 1 \leq x < 100$.

Step 1 Let a_1 be the first digit of the square root of x.

Step 2 Calculate

$$a_2 = \frac{a_1}{2} + \frac{x}{2a_1}.$$

Step 3 Repeat Step 2

$$a_n = \frac{a_{n-1}}{2} + \frac{x}{2a_{n-1}}$$

until the desired precision is reached.

Problem 11. Use Algorithm 4 to find an approximate value of $\sqrt{3}$ with 3 decimal place precision.

Problem 12*. Explain why Newton's method works.

Problem 13*. Try to find a different algorithm for approximating the square root of a number between 1 and 100. Name the algorithm with your own name.

4.2 Algorithm to find the cubic root

Here is Newton's method of finding the cubic root of a number. The ancient Chinese way described in the book *Nine Chapters* is quite complicated. It is a generalization of Algorithm 1. You can perhaps try to rediscover it yourself.

4.2.1 Newton's method of finding the cubic root of x

We assume $1 \leq x < 1000$.

Step 1 Let a_1 be the first digit of the cubic root of x.

Step 2 Calculate

$$a_2 = \frac{2a_1}{3} + \frac{x}{3a_1^2}.$$

Step 3 Repeat Step 2

$$a_n = \frac{2a_{n-1}}{3} + \frac{x}{3a_{n-1}^2}$$

until the desired precision is reached.

You perhaps can guess now that Newton's method should work in finding any root of a positive number x. Indeed, Newton's method can be used to find an approximate solution to any equation. It is one of the most useful method in modern numerical computation. Its general formula requires a simple use of a calculus concept, called the *derivative.*

Problem 14. Use Newton's method to find the first 4 significant digits of the cubic root of 3.

4.3 Finding any root with bisection method

Is there a way to find an approximate value of a solution to an equation such as

$$x^x = 3?$$

The answer is yes. But you may need to use your calculator a little bit.

Step 1 Estimate the solution: Find an interval $[a, b]$ so that $b^b > 3$ and $a^a < 3$.

Step 2 Try the midpoint of the interval now: calculate $(\frac{a+b}{2})^{(\frac{a+b}{2})}$. If the number is less than 3, call it a new a. If the number is greater than 3, call it a new b.

Step 3 Repeat Step 2 until the desired precision is reached.

Problem 15. Use the bisection method to find an approximate value of the solution to the equation

$$x^x = 3$$

accurate up to two decimal places.