Chapter 3

Millet and Rice

This chapter is about exchange rates. Instead of grains in the original text, we will be using currencies. The problems in this chapter are again for you to practice operations on fractions. So, express your results in common fractions only.

Assume these exchange rates between three currencies in Year 2000 and Year 2006 (Actual exchange rates vary during a year. Go to http://www.exchangerate.com to find daily exchange rates. For gold price, search Google).

Year 2000:

10 Euros = 9 US dollars; 1 Ounce gold = US 280 dollar.

Year 2006:

3 Euros = 4 US dollars; 1 Ounce gold = 450 Euros.

Problem 1. A round trip airline ticket from Greensboro to Paris cost about $720 in Year 2000. Assuming that airline ticket prices have remained approximately at the same level in terms of Euros during the past years. Estimate the price of a round trip airline ticket from Greensboro to Paris in Year 2006 in US dollars.

Problem 2 If Tom bought 1000 US dollar worth of Gold in Year 2000, how many US dollars could he get if he sold the Gold back in Year 2006?

What was the percentage of increase in the value of his money from Year 2000 to Year 2006?

Chapter 4

Uneven Distribution

Find solutions to all problems without using a calculator, unless it is noted otherwise. Express your results in common fractions.

The Proportional Distribution Rule

> *Lay down the rates for distribution, then, add; take the sum as divisor; multiply the amount to be distributed by each rate as dividends.* $\cdots\cdots$

Problem 1. A family of 3 went holiday shopping. The father spent half as much as the wife, the wife spent half as much as the teenage child. They spent (charged on their credit card!) total $ 1400. How much did each of them spend?

Problem 2. There are 5 officials with ranks 5,4,3,2, and 1 who went hunting and got 5 deers. How many should each get if the deer is to be divided proportional to their ranks?

Problem 3. Reconsider the previous problem if there are 9 officials ranked 9,8,7,6,5,4,3, 2, and 1.

Problem 4. For any natural number n, find the formulas for the following sums:

(1) $1 + 2 + 3 + \cdots + n =$

(2) $3 + 6 + 9 + \cdots + 3n =$

(3) $4 + 7 + 10 + \cdots + (3n + 1) =$

A sequence of numbers, $a_1, a_2, \cdots, a_n$ *is called an* **arithmetic progression** *is the difference between every pair of consecutive numbers in the sequence is the same:*

$$d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}.$$

Problem 5. Show that for an arithmetic progression $a_1, a_2, \cdots, a_n$, the sum is given by

$$a_1 + a_2 + \cdots + a_n = \frac{a_1 + a_n}{2} n.$$

Or equivalently,

$$a_1 + a_2 + \cdots + a_n = \frac{a_1 + (a_1 + (n-1)d)}{2} n = a_1 n + \frac{(n-1)n}{2} d.$$

Problem 6. The same 5 officials in Problem 2 now need to pay a total of 100 coins of taxes. The tax code says that each should pay an amount inversely proportional to his or her rank. How many coins should each pay?

Problem 7. Reconsider Problem 6 with 9 officials with ranks given in Problem 3.

The sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$ *is called the* **Harmonic sequence**.

Problem 8. Find the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{12}$ in common fraction.

Problem 9. Tell why the following inequalities hold

(1) $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$

(2) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2}$

(3) $\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{2}$

$\cdots\cdots$

(4) $\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n} > \frac{1}{2}$

Conclude that
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n} > \frac{n}{2}.$$

What can you say about the sum of the harmonic sequence when n gets bigger and bigger?

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}.$$

A sequence $a_1, a_2, a_3, \cdots, a_n$ *is called a* **Geometric progression** *if the ratio between every consecutive numbers is a constant* r*:*

$$r = \frac{a_2}{a_1} = \cdots = \frac{a_n}{a_{n-1}}.$$

Or,
$$a_n = ra_{n-1} = r^2 a_{n-2} = \cdots = r^{n-1} a_1.$$

Problem 10. Let S be the sum of a geometric progression

$$S = 1 + r + r^2 + \cdots + r^{n-1}.$$

Verify the following identities:

(1) $rS - S = r^n - 1$

(2) $1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r-1} = \frac{1-r^n}{1-r}$

Problem 11. A skillful weaver who doubles her product every day. In 5 days, she finishes weaving 5 meters. How much does she weave in the first, second, third, fourth, and fifth day? If in 10 days, she finished 10 meters, how much does she weave in each successive day? (You may wish to generalize this problem: if she weaves n meters in m days, how much does she weave in each successive day?)

Problem 12. A mouse is digging a hole in a thick wall of a castle. His progress is slowed 50 percent with each passing day (you can imagine it is much harder to dig when he gets deeper into the wall). In 5 days, he digs 4 meters, how much does he dig in each successive day? If the wall is 5 meters thick, can the mouse ever dig through the wall?

Problem 13. A big mouse was eating one pound of cheese in the shape of an equilateral triangle. On the first day, he ate $\frac{1}{4}$ of it; on the second day, he ate $\frac{1}{4}$ of what was left from the first day: the amount he eats during the day is always $\frac{1}{4}$ of what he has at the beginning of the day. How much will he have eaten by the end of the 10th day? Will he ever be able to finish the entire piece of cheese?

If you wish to see how he eats his cheese geometrically, go to http://www.shodor.org/interactivate/activities/SierpinskiCarpet/ to find the Sierpinski triangle. Can you tell what is so special about this triangle?

Problem 14. For the same mouse eating cheese problem, assume now that the amount the mouse eats during the day is always equal to $\frac{1}{9}$ of what he has at the beginning of the day. Find the amount of leftover after n days.

Problem 15. There was a family in ancient China whose house was blocked by a huge mountain. The father, whose name was Yu Gong (Mr. Dumb), called a family meeting and decided to remove the mountain. A traveler named Zhi Sou (Mr. Smart) saw what Yu Gong was doing and laughed at his effort and said his family could never be able to remove the mountain. Yu Gong relied : " When I die, my sons will continue, and my sons will have their sons. The mountain is not growing. Can you tell me why the mountain can not be removed?"

Assume that each man in the family will have at least two sons. Each man can dig the mountain for 30 years in his life time and removes about 300 cubic meters of rocks each year. Assume they need to remove $1000 \times 1000 \times 3000$ cubic meters of rocks. How many generations will it take to finish the work? (A calculator would be helpful.)

(The story ended in a different way: God was moved by Yu Gong's determination and removed the mountain for the family.)

Interest Rate: *When the annual interest rate is* 6 *percent, if you deposit* \$100 *in the bank, you get* \$106 *back after an entire year. This is why this* 6 *percent rate is also called the* **return rate.** *If you borrow* \$100, *you need to pay back* \$106 *after a year.*

Compound Interest: *Banks normally do not calculate interests yearly. The interest is often compounded monthly for loans. For example, if you borrow* \$100 *at the rate* 6%, *the monthly rate will be* $0.5 = (6/12)\%$. *At the end of the first month, you will owe the bank,* $100 \times (1 + 0.005) = 100.5$ *dollars. At the end of the second month, you will owe the bank* $100.5 \times (1 + 0.005) = 100 \times (1 + 0.005)^2 = 101.0025$ *dollar. By the end of the first year, the total amount, after rounding to the nearest penny, the amount you owe the bank becomes*

$$100 \times (1 + 0.005)^{12} = 106.17.$$

Notice that the amount is slightly bigger than 106. *If you deposit* \$100 *dollars at the annual interest rate* 6% *and the interest is compounded monthly, you will get* \$106.17. *The* 6.17% *return is also called the annual* **rate of yield.**

Problem 16. (Retirement planning) Richard has a retirement account with a mutual fund company. Every month, he deposits \$500 dollars. The annual return rate is 6 percent. The monthly return rate would then be $0.5(= 6/12)$ percent. How much money will he have at the time when he retires after 20 years? (Hint: setup it as the sum of a geometric progression. A calculator is needed.)

Problem 17 (Mortgage calculation) Jack needs to borrow 14,000 dollars from his credit union to buy a new car. The (annual) interest rate is 3.6%. Thus, the monthly interest rate is 0.3%. The term of the loan is 2 years. That means, he needs to pay the loan off within 24 months. What should be his monthly payment? (Hint: Because a monthly payment can not include a fraction of a penny, the last payment is usually different from the regular monthly payment. You will need to try to make the last payment as close to, but not bigger than, the regular payment as possible. Need to use a calculator intelligently to find the result.)

4.1 Additional problems for mathematically more adventurous persons

In the Pascal's triangle, the numbers are labeled according to their locations. The rows are counted starting from **zero** and so are the positions in each row. For example, the number in the row 3 and at the position 2 from left is denoted by $\binom{3}{2} = 3$. The easiest way to construct the triangle is the following:

(1) For any row number $n \geq 0$, $\binom{n}{0} = \binom{n}{n} = 1$.

(2) For any row number $n \geq 2$ and $1 \leq k < n$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Problem 1. Show that the following relations hold.

(1) $\binom{n}{1} = \binom{n}{n-1} = n$, for all $n \geq 1$.

(2) $\binom{n}{k} = \binom{n}{n-k} = n$, for all $n \geq 1$, $0 \leq k \leq n$.

(3) $\binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \cdots + \binom{n}{0} = \binom{n+1}{1} = n + 1$.

(4) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

(5) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$.

(6) For any $r \geq 2$, $\binom{r+1}{2} + \binom{r}{2} = r^2$. This equality holds for $r = 1$ if we define $\binom{1}{2} = 0$.

(7) For $n \geq 3$, $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$.

(8) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(9) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Problem 2. In the Pascal's triangle, paint the locations with odd numbers black, you get the Sierpinski triangle. You will need to have at least 16 rows to see this effect. Can you tell why we get a Sierpinski triangle?

Problem 3. The number $\binom{n}{k}$ is also called the combination number: from n choosing k.

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2 \cdot 1} = \frac{n!}{(n-k)!k!}.$$

Problem 4. Find the sums (no calculators needed!).

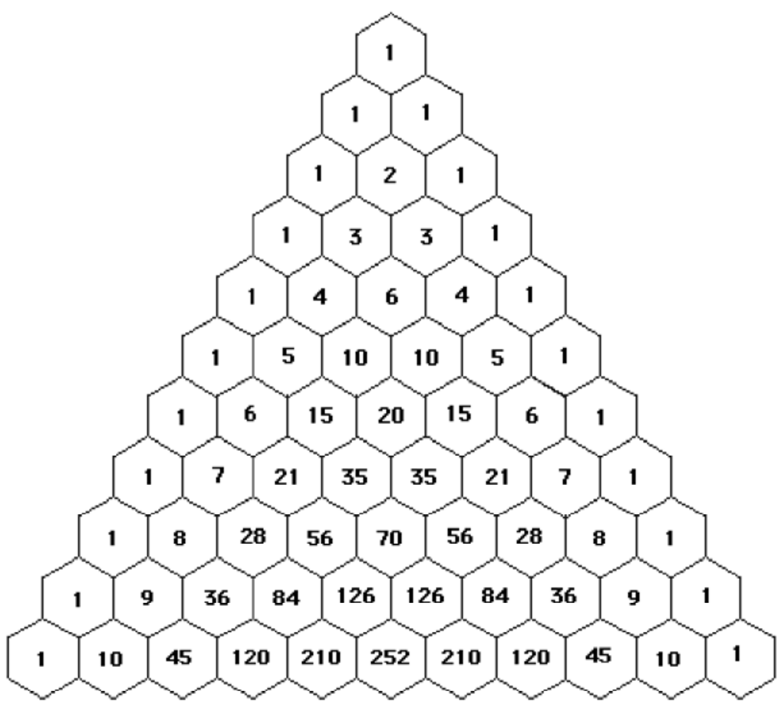
(1) $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{90} =$

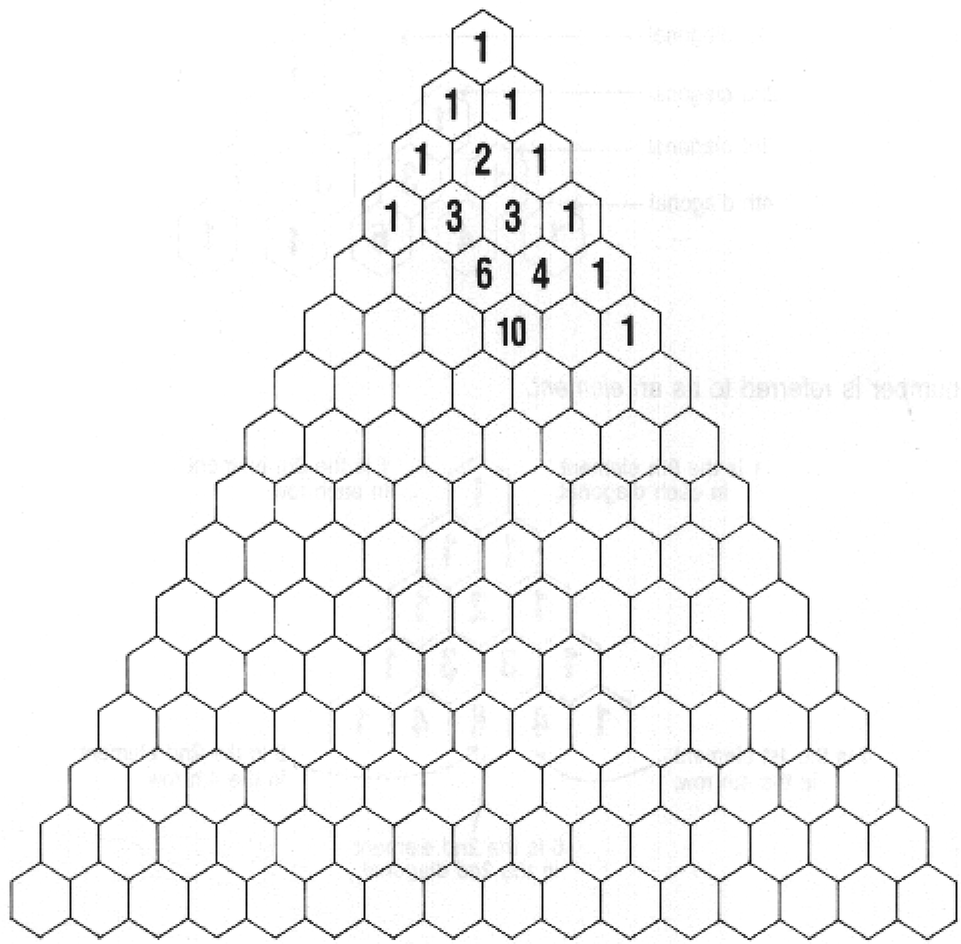
(2) $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{9900} =$

(3) $1 + \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots + \frac{1}{n\times(n+1)} =$

Problem 5. Show that the following inequality holds for any whole number n.

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \frac{1}{n^2} < 2.$$

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