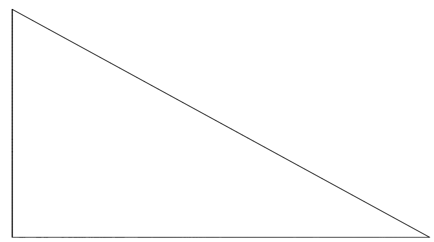
2.3 Pythagorean Theorem and the value of π

For a right angle triangle, the square of the hypotenuse equals the sum of the squares of the sides:

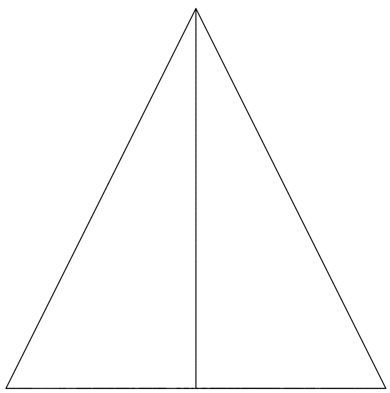
$$a^2 + b^2 = c^2.$$

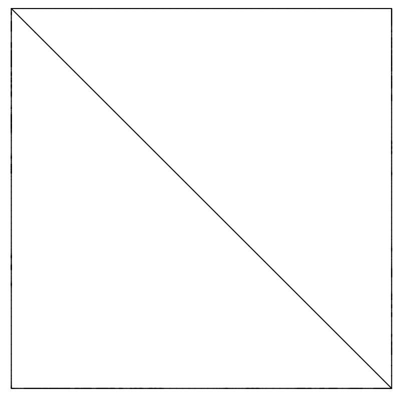


The Chinese name for this theorem is *Gou Gu* theorem. Its applications constitute the last chapter of *Nine Chapters*. But it is used by Liu Hui in the first chapter to justify the area formula for a disk. The first 6 problems of Section 2.5 give an outline of Liu Hui's argument and his calculation of $\pi = 3.14159265358979323846264338327950288419716939937510582097494 459230781640628620899862803482534211706 8\cdots$.

2.4 Stories of two triangles

Have you ever wondered why the triangles you bought for your geometry class come in only two shapes? Why are these two triangles so special? To find out the possible reason, you need two sets of them. If you put two of the same kind together, you get one equilateral triangle and one equilateral quadrilateral, the square. Because one triangle is half of an equilateral triangle, the shorter side is exactly half of the hypotenuse. If we have the shorter side 1, then the hypotenuse must be 2. By Pythagorean theorem, the other side is $\sqrt{3}$. We now know that the angles are $30^o, 60^o$, and 90^o, respectively. But 2200 years ago, Chinese did not have such way of measuring angles. The way they described the angles was the way we use now in Calculus: in radian. They used the arc length facing the angle to measure it.





We do not need to use the fact that the smaller acute angle is 30^o to conclude that the side opposite to this angle is equal to half of the hypotenuse. We only need to know that the smaller acute angle is half of that of an equilateral triangle. Then, we can use the similar triangle argument to get the same conclusion: *in a right angle triangle, if the smaller acute angle is half of the angle of an equilateral triangle, then, the opposite side of this smaller acute angle is half of the hypotenuse.*

2.5 Problems

In these problems, the circle is assumed to be the unit circle, a circle with radius one. Express your exact results of Problems 1 to 5 in the simplest radical form.

1. Find the area of an equilateral triangle inscribed in the unit circle.

2. Find the length of the side of a regular hexagon inscribed in the circle.

3. Find the area of a regular hexagon inscribed in the circle.

4. Find the area of a regular dodecagon (12 sides) inscribed in a unit circle.

5. Find the area of a regular 24-gon inscribed in a unit circle.

6. Continue calculating the area of the regular 3×2^n-gon inscribed in a unit circle for $n = 4, 5, 6, \cdots$ until your are tired of doing it. You may express your approximate results in decimal forms now using a calculator. Liu Hui did the calculation of $3 \times 2^9 = 1536$-gon's area in decimal form. If you know how to program your calculator, find the smallest value of n such that the area of the 3×2^n-gon is greater than 3.14159.

7. Assume that the area of the regular 3×2^n-gon inscribed in a circle is A_n, find the relation between A_n and A_{n+1}.

8. Start with the area of the inscribed square in a unit circle. Continue with the areas of the regular octagon, 16-gon, etc.. Compare your results with previous calculations.

9*. Show that the area of a disk is equal to the product of half circumference times half diameter. (This is the formula given in *Nine Chapters.* Liu Hui justified the formula with an infinite process of approximating the area using regular polygons).

10*. Again, assume that the area of the regular 3×2^n-gon inscribed in a circle is A_n, determine the order of of $\pi - A_n$ in terms of $\frac{1}{n}$. (It should be rather easy if you use trigonometry and a little bit of calculus. But trigonometry was not available at that time.)

11*. Is there another shape in the plane such that the area is equal to half circumference times half diameter? The diameter of a shape is the longest distance between two points inside this shape.

Hint: this is a rather hard problem. There is a simple proof if you can find and understand the *Cauchy-Crofton formula* and the *Isoperimetric Inequality.*

12*. Half circumference seems to be a magic number. Prove the following Heron's theorem: the area of a triangle with sides a, b, c is equal to $\sqrt{s(s-a)(s-b)(s-c)}$ where s is the half circumference.

13. You may ask why people did not use the circumscribed regular polygons to approximate the area of a disk? You should definitely try and see what you get.

14*. Liu Hui tried to use the same method to find the area of the sphere. But he did not succeed. Try it yourself to see if you can get anything interesting. Can you find the surface areas of regular polyhedrons inscribed in a unit ball (the ball with radius one)? Search the internet to find the names of all regular polyhedrons.

15. How about finding the volume of a ball? What is the relation between the surface area of the ball and the volume of the ball? We will revisit this problem in a later chapter when we deal with volumes.