New Commentaries on Nine Chapters of Mathematical Arts

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May 24, 2007

Chapter 0

Introduction

It is surprising how fruitful it can be to read a book written more than 2000 years ago on mathematics. Majority of problems collected in the book *Nine Chapters of Mathematical Arts* (NCMA), which is believed to be written around 200 B.C., are still very suitable for today's classrooms. However, unlike today's mathematical textbooks that are often loaded with definitions, NCMA and the commentaries therein focused exclusively on algorithms for solving practical problems where definitions are either not necessary or intuitively obvious. Students are thus, exposed to rigorous calculations before abstraction. These problems are still suitable for today's secondary school students who often know many modern mathematical concepts and yet lack the training in problem solving skills.

Even though the problems compiled in this booklet follow the schemes presented in NCMA, they are not exactly the same problems. Many problems are inspired by those in NCMA and they represent part of today's core knowledge of elementary mathematics.

The level of difficulty of these problem varies from very easy to very hard. Some are suitable for a college mathematics major to do a project on. There are also problems that I, at the moment, do not have a solution. The problems are compiled without extensive research on the references. A problem I do not know how to solve at the moment does not mean it is still an open problem.

It is true that certain methods used in the book need justification. Such a task is often difficult for young students. The book NCMA rarely gave proofs for its methods. Many calculations were later justified by Liu Hui in his commentaries written around 263 A.D. Mathematically rigorous solutions to most of these problems will require axiomatic treatment which is what we want to avoid here. Thus, the understanding of basic concepts such as the length of a segment, the height of a triangle, etc. is based on our intuition, or common notion. Such concepts are now often taught at elementary schools.

Chapter 1

Rectangular Field

1.1 Area

The area of a rectangle with a base width of a steps and a depth of b steps is $a \times b$ *product steps*.

1.2 Problems

1. Show that the area of a parallelogram is equal to the product of the base and the height by using the cut-and-paste method.

2. Show that the area of a triangle is equal to half of the product of the base and the height.

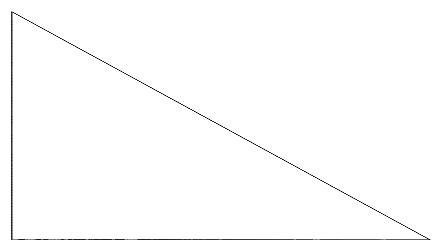
3. Show that the area of a trapezoid is equal to half of the product of the sum of two parallel bases and the height:

$$A = \frac{a+b}{2} \times h.$$

1.3 Pythagorean Theorem and the value of π

For a right angle triangle, the square of the hypotenuse equals the sum of the squares of the sides:

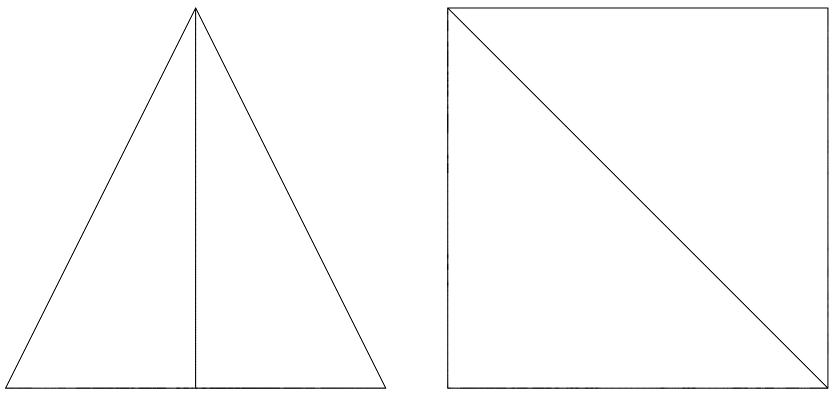
$$a^2 + b^2 = c^2.$$



The Chinese name for this theorem is *Gou Gu* theorem. Its applications constitute the last chapter of *Nine Chapters*. But it is used by Liu Hui in the first chapter to justify the area formula for a disk. The first 6 problems of Section 1.5 give an outline of Liu Hui's argument and his calculation of $\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706 8\cdots$.

1.4 Stories of two triangles

Have you ever wondered why the triangles you bought for your geometry class come in only two shapes? Why are these two triangles so special? To find out the possible reason, you need two sets of them. If you put two of the same kind together, you get one equilateral triangle and one equilateral quadrilateral, the square. Because one triangle is half of an equilateral triangle, the shorter side is exactly half of the hypotenuse. If we have the shorter side 1, then the hypotenuse must be 2. By Pythagorean theorem, the other side is $\sqrt{3}$. We now know that the angles are $30^o, 60^o$, and 90^o, respectively. But 2200 years ago, Chinese did not have such way of measuring angles. The way they described the angles was the way we use now in Calculus: in radian. They used the arc length facing the angle to measure it.



We do not need to use the fact that the smaller acute angle is 30^o to conclude that the side opposite to this angle is equal to half of the hypotenuse. We only need to know that the smaller acute angle is half of that of an equilateral triangle. Then, we can use the similar triangle argument to get the same conclusion: *in a right angle triangle, if the smaller acute angle is half of the angle of an equilateral triangle, then, the opposite side of this smaller acute angle is half of the hypotenuse.*

1.5 Problems

In these problems, the circle is assumed to be the unit circle, a circle with radius one. Express your exact results of Problems 1 to 5 in the simplest radical form.

1. Find the area of an equilateral triangle inscribed in the unit circle.

2. Find the length of the side of a regular hexagon inscribed in the circle.

3. Find the area of a regular hexagon inscribed in the circle.

4. Find the area of a regular dodecagon (12 sides) inscribed in a unit circle.

5. Find the area of a regular 24-gon inscribed in a unit circle.

6. Continue calculating the area of the regular 3×2^n-gon inscribed in a unit circle for $n = 4, 5, 6, \cdots$ until your are tired of doing it. You may express your approximate results in decimal forms now using a calculator. Liu Hui did the calculation of $3 \times 2^9 = 1536$-gon's area in decimal form. If you know how to program your calculator, find the smallest value of n such that the area of the 3×2^n-gon is greater than 3.14159.

7. Assume that the area of the regular 3×2^n-gon inscribed in a circle is A_n, find the relation between A_n and A_{n+1}.

8. Start with the area of the inscribed square in a unit circle. Continue with the areas of the regular octagon, 16-gon, etc.. Compare your results with previous calculations.

9*. Show that the area of a disk is equal to the product of half circumference times half diameter. (This is the formula given in *Nine Chapters*. Liu Hui justified the formula with an infinite process of approximating the area using regular polygons).

10*. Again, assume that the area of the regular 3×2^n-gon inscribed in a circle is A_n, determine the order of of $\pi - A_n$ in terms of $\frac{1}{n}$. (It should be rather easy if you use trigonometry and a little bit of calculus. But trigonometry was not available at that time.)

11*. Is there another shape in the plane such that the area is equal to half circumference times half diameter? The diameter of a shape is the longest distance between two points inside this shape.

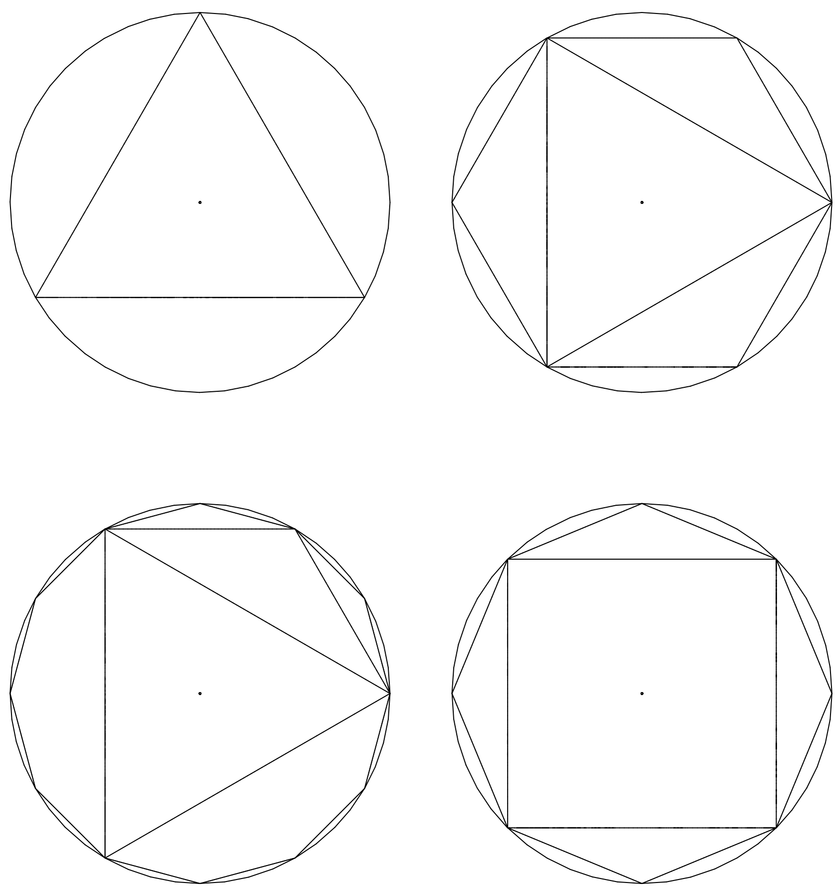
Hint: this is a rather hard problem. There is a simple proof if you can find and understand the *Cauchy-Crofton formula* and the *Isoperimetric Inequality.*

12*. Half circumference seems to be a magic number. Prove the following Heron's theorem: the area of a triangle with sides a, b, c is equal to $\sqrt{s(s-a)(s-b)(s-c)}$ where s is the half circumference.

13. You may ask why people did not use the circumscribed regular polygons to approximate the area of a disk? You should definitely try and see what you get.

14*. Liu Hui tried to use the same method to find the area of the sphere. But he did not succeed. Try it yourself to see if you can get anything interesting. Can you find the surface areas of regular polyhedrons inscribed in a unit ball (the ball with radius one)? Search the internet to find the names of all regular polyhedrons.

15. How about finding the volume of a ball? What is the relation between the surface area of the ball and the volume of the ball? We will revisit this problem in a later chapter when we deal with volumes.



More isoperimetric problems

More area maximizing problems

Chapter 2

Millet and Rice

This chapter is about exchange rates. Instead of grains in the original text, we will be using currencies. The problems in this chapter are again for you to practice operations on fractions. So, express your results in common fractions only.

Assume these exchange rates between three currencies in Year 2000 and Year 2006 (Actual exchange rates vary during a year. Go to http://www.exchangerate.com to find daily exchange rates. For gold price, search Google).

Year 2000:

10 Euros = 9 US dollars; 1 Ounce gold = US 280 dollar.

Year 2006:

3 Euros = 4 US dollars; 1 Ounce gold = 450 Euros.

Problem 1. A round trip airline ticket from Greensboro to Paris cost about $720 in Year 2000. Assuming that airline ticket prices have remained approximately at the same level in terms of Euros during the past years. Estimate the price of a round trip airline ticket from Greensboro to Paris in Year 2006 in US dollars.

Problem 2 If Tom bought 1000 US dollar worth of Gold in Year 2000, how many US dollars could he get if he sold the Gold back in Year 2006?

What was the percentage of increase in the value of his money from Year 2000 to Year 2006?

Chapter 3

Uneven Distribution

Find solutions to all problems without using a calculator, unless it is noted otherwise. Express your results in common fractions.

The Proportional Distribution Rule

> *Lay down the rates for distribution, then, add; take the sum as divisor; multiply the amount to be distributed by each rate as dividends.* $\cdots\cdots$

Problem 1. A family of 3 went holiday shopping. The father spent half as much as the wife, the wife spent half as much as the teenage child. They spent (charged on their credit card!) total $ 1400. How much did each of them spend?

Problem 2. There are 5 officials with ranks 5,4,3,2, and 1 who went hunting and got 5 deers. How many should each get if the deer is to be divided proportional to their ranks?

Problem 3. Reconsider the previous problem if there are 9 officials ranked 9,8,7,6,5,4,3, 2, and 1.

Problem 4. For any natural number n, find the formulas for the following sums:

(1) $1 + 2 + 3 + \cdots + n =$

(2) $3 + 6 + 9 + \cdots + 3n =$

(3) $4 + 7 + 10 + \cdots + (3n + 1) =$

A sequence of numbers, $a_1, a_2, \cdots, a_n$ *is called an* **arithmetic progression** *is the difference between every pair of consecutive numbers in the sequence is the same:*

$$d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}.$$

Problem 5. Show that for an arithmetic progression $a_1, a_2, \cdots, a_n$, the sum is given by

$$a_1 + a_2 + \cdots + a_n = \frac{a_1 + a_n}{2} n.$$

Or equivalently,

$$a_1 + a_2 + \cdots + a_n = \frac{a_1 + (a_1 + (n-1)d)}{2} n = a_1 n + \frac{(n-1)n}{2} d.$$

Problem 6. The same 5 officials in Problem 2 now need to pay a total of 100 coins of taxes. The tax code says that each should pay an amount inversely proportional to his or her rank. How many coins should each pay?

Problem 7. Reconsider Problem 6 with 9 officials with ranks given in Problem 3.

The sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$ *is called the* **Harmonic sequence**.

Problem 8. Find the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{12}$ in common fraction.

Problem 9. Tell why the following inequalities hold

(1) $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$

(2) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2}$

(3) $\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{2}$

$\cdots\cdots$

(4) $\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n} > \frac{1}{2}$

Conclude that
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n} > \frac{n}{2}.$$

What can you say about the sum of the harmonic sequence when n gets bigger and bigger?

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}.$$

A sequence $a_1, a_2, a_3, \cdots, a_n$ *is called a* **Geometric progression** *if the ratio between every consecutive numbers is a constant* r*:*

$$r = \frac{a_2}{a_1} = \cdots = \frac{a_n}{a_{n-1}}.$$

Or,
$$a_n = ra_{n-1} = r^2 a_{n-2} = \cdots = r^{n-1} a_1.$$

Problem 10. Let S be the sum of a geometric progression

$$S = 1 + r + r^2 + \cdots + r^{n-1}.$$

Verify the following identities:

(1) $rS - S = r^n - 1$

(2) $1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n-1}{r-1} = \frac{1-r^n}{1-r}$

Problem 11. A skillful weaver who doubles her product every day. In 5 days, she finishes weaving 5 meters. How much does she weave in the first, second, third, fourth, and fifth day? If in 10 days, she finished 10 meters, how much does she weave in each successive day? (You may wish to generalize this problem: if she weaves n meters in m days, how much does she weave in each successive day?)

Problem 12. A mouse is digging a hole in a thick wall of a castle. His progress is slowed 50 percent with each passing day (you can imagine it is much harder to dig when he gets deeper into the wall). In 5 days, he digs 4 meters, how much does he dig in each successive day? If the wall is 5 meters thick, can the mouse ever dig through the wall?

Problem 13. A big mouse was eating one pound of cheese in the shape of an equilateral triangle. On the first day, he ate $\frac{1}{4}$ of it; on the second day, he ate $\frac{1}{4}$ of what was left from the first day: the amount he eats during the day is always $\frac{1}{4}$ of what he has at the beginning of the day. How much will he have eaten by the end of the 10th day? Will he ever be able to finish the entire piece of cheese?

If you wish to see how he eats his cheese geometrically, go to http://www.shodor.org/interactivate/activities/SierpinskiCarpet/ to find the Sierpinski triangle. Can you tell what is so special about this triangle?

Problem 14. For the same mouse eating cheese problem, assume now that the amount the mouse eats during the day is always equal to $\frac{1}{9}$ of what he has at the beginning of the day. Find the amount of leftover after n days.

Problem 15. There was a family in ancient China whose house was blocked by a huge mountain. The father, whose name was Yu Gong (Mr. Dumb), called a family meeting and decided to remove the mountain. A traveler named Zhi Sou (Mr. Smart) saw what Yu Gong was doing and laughed at his effort and said his family could never be able to remove the mountain. Yu Gong relied : " When I die, my sons will continue, and my sons will have their sons. The mountain is not growing. Can you tell me why the mountain can not be removed?"

Assume that each man in the family will have at least two sons. Each man can dig the mountain for 30 years in his life time and removes about 300 cubic meters of rocks each year. Assume they need to remove $1000 \times 1000 \times 3000$ cubic meters of rocks. How many generations will it take to finish the work? (A calculator would be helpful.)

(The story ended in a different way: God was moved by Yu Gong's determination and removed the mountain for the family.)

Interest Rate: *When the annual interest rate is* 6 *percent, if you deposit* \$100 *in the bank, you get* \$106 *back after an entire year. This is why this* 6 *percent rate is also called the* **return rate.** *If you borrow* \$100, *you need to pay back* \$106 *after a year.*

Compound Interest: *Banks normally do not calculate interests yearly. The interest is often compounded monthly for loans. For example, if you borrow* \$100 *at the rate* 6%*, the monthly rate will be* $0.5 = (6/12)\%$*. At the end of the first month, you will owe the bank,* $100 \times (1 + 0.005) = 100.5$ *dollars. At the end of the second month, you will owe the bank* $100.5 \times (1 + 0.005) = 100 \times (1 + 0.005)^2 = 101.0025$ *dollar. By the end of the first year, the total amount, after rounding to the nearest penny, the amount you owe the bank becomes*

$$100 \times (1 + 0.005)^{12} = 106.17.$$

Notice that the amount is slightly bigger than 106*. If you deposit* \$100 *dollars at the annual interest rate* 6% *and the interest is compounded monthly, you will get* \$106.17*. The* 6.17% *return is also called the annual* **rate of yield.**

Problem 16. (Retirement planning) Richard has a retirement account with a mutual fund company. Every month, he deposits \$500 dollars. The annual return rate is 6 percent. The monthly return rate would then be $0.5(= 6/12)$ percent. How much money will he have at the time when he retires after 20 years? (Hint: setup it as the sum of a geometric progression. A calculator is needed.)

Problem 17 (Mortgage calculation) Jack needs to borrow 14,000 dollars from his credit union to buy a new car. The (annual) interest rate is 3.6%. Thus, the monthly interest rate is 0.3%. The term of the loan is 2 years. That means, he needs to pay the loan off within 24 months. What should be his monthly payment? (Hint: Because a monthly payment can not include a fraction of a penny, the last payment is usually different from the regular monthly payment. You will need to try to make the last payment as close to, but not bigger than, the regular payment as possible. Need to use a calculator intelligently to find the result.)

3.1 Additional problems for mathematically more adventurous persons

In the Pascal's triangle, the numbers are labeled according to their locations. The rows are counted starting from **zero** and so are the positions in each row. For example, the number in the row 3 and at the position 2 from left is denoted by $\binom{3}{2} = 3$. The easiest way to construct the triangle is the following:

(1) For any row number $n \geq 0$, $\binom{n}{0} = \binom{n}{n} = 1$.

(2) For any row number $n \geq 2$ and $1 \leq k < n$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Problem 1. Show that the following relations hold.

(1) $\binom{n}{1} = \binom{n}{n-1} = n$, for all $n \geq 1$.

(2) $\binom{n}{k} = \binom{n}{n-k} = n$, for all $n \geq 1$, $0 \leq k \leq n$.

(3) $\binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \cdots + \binom{n}{0} = \binom{n+1}{1} = n + 1$.

(4) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

(5) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$.

(6) For any $r \geq 2$, $\binom{r+1}{2} + \binom{r}{2} = r^2$. This equality holds for $r = 1$ if we define $\binom{1}{2} = 0$.

(7) For $n \geq 3$, $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$.

(8) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(9) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Problem 2. In the Pascal's triangle, paint the locations with odd numbers black, you get the Sierpinski triangle. You will need to have at least 16 rows to see this effect. Can you tell why we get a Sierpinski triangle?

Problem 3. The number $\binom{n}{k}$ is also called the combination number: from n choosing k.

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2 \cdot 1} = \frac{n!}{(n-k)!k!}.$$

Problem 4. Find the sums (no calculators needed!).

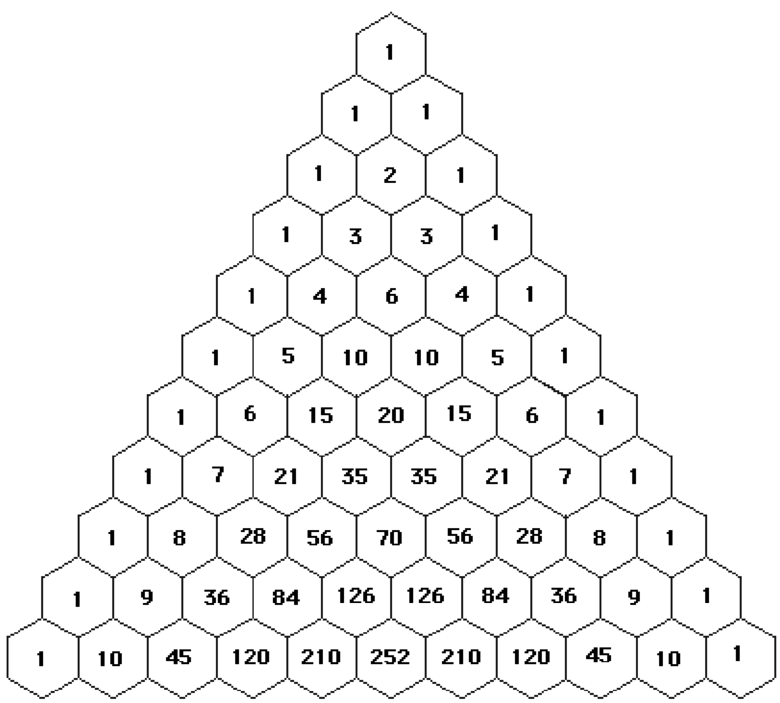
(1) $1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots+\frac{1}{90}=$

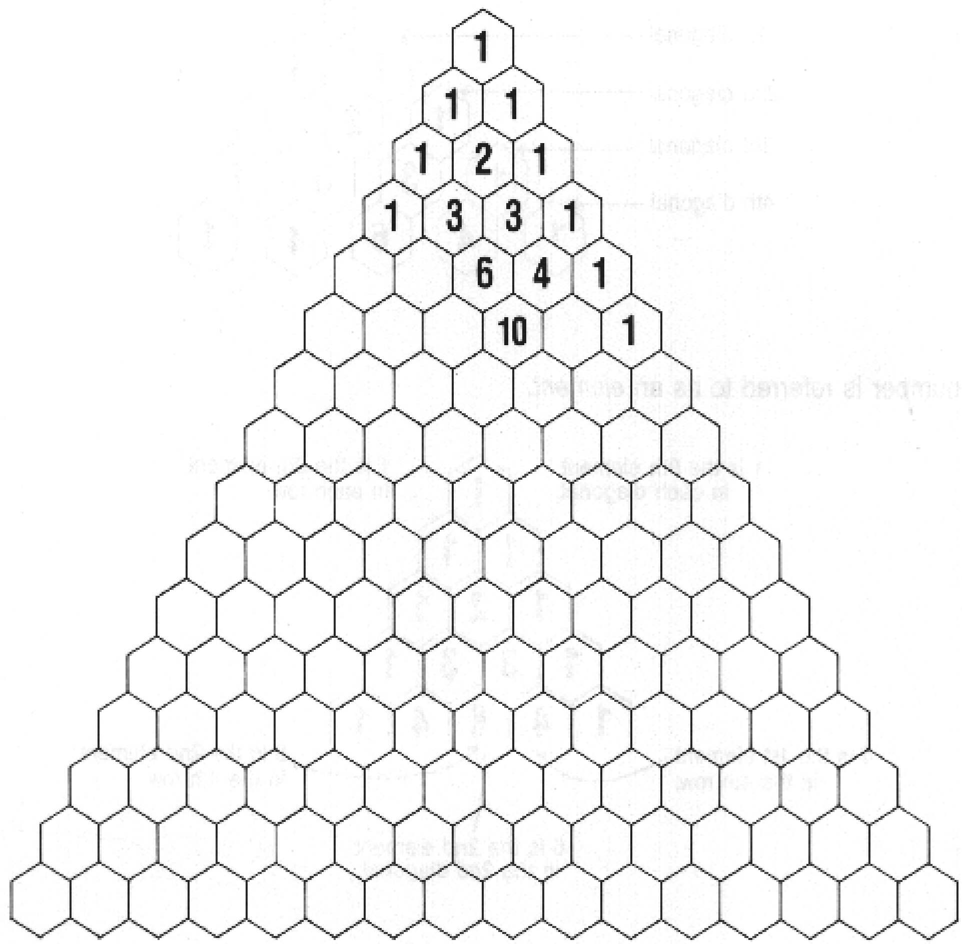
(2) $1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots+\frac{1}{9900}=$

(3) $1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{n\times(n+1)}=$

Problem 5. Show that the following inequality holds for any whole number n.

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots+\frac{1}{n^2}<2.$$




1
1 1
1 2 1
1 3 3 1
6 4 1
10 1

Chapter 4

Shortening the Width

For convenience, the units have been changed.

Problem 1. 1 acre = 4840 square yards.

(1) Given a rectangular field whose width is $1\frac{1}{2}$ yards. Assume the area is 1 acre. What is its length?

(2) Given a rectangular field whose width is $1 + \frac{1}{2} + \frac{1}{3}$ yards. Assume the area is 1 acre. What is its length?

Problem 2. Find the first significant (non-zero) digit of the following square roots. Do not use a calculator.

(1) $\sqrt{1.919}$, $\sqrt{3.919}$, $\sqrt{4.0919}$, $\sqrt{13.919}$, $\sqrt{23.919}$, $\sqrt{73.0919}$, $\sqrt{99.919}$.

(2) $\sqrt{0.1919}$, $\sqrt{1.919}$, $\sqrt{19.19}$, $\sqrt{191.9}$, $\sqrt{1919.00}$, $\sqrt{19190}$, $\sqrt{191900}$.

Problem 3. Find the first significant (non-zero) digit of the following cubic roots. Do not use a calculator.

$$\sqrt[3]{1.919},\ \sqrt[3]{4.0919},\ \sqrt[3]{13.919},\ \sqrt[3]{239.19},\ \sqrt[3]{730.919},\ \sqrt[3]{999.19}.$$

4.1 Finding an approximate value of a square root

There are numerous ways to find an approximate value of a square root. In the book *Nine Chapters*, the following method is used.

4.1.1 Algorithm 1: Ancient Chinese Way

Given a number x. Assume $1 \leq x < 100$. If not, either divide or multiply x by $100, 10000$ etc. to bring the number into the interval. After finding the approximate value of the square root of the new number, divide or multiply it by $10, 100$ etc.

Step 1 Find the first digit of the square root and record it as a.

Step 2 Subtract from x the square of a, a^2, the result is called $y = x - a^2$.

Step 3 Multiply y by 100. Find another digit b so that

$$(20a + b)b \leq 100y < (20a + b + 1)(b + 1).$$

Subtract $(20a + b)b$ from $100y$.

Step 4 If the result from Step 3 is 0, terminate the process. The square root is $a + b/10 = a.b$.

If the result is not 0, it is called a new $\widetilde{y}$ and repeat Step 3 with a new $\tilde{a} = 10a + b$.

The approximate value of the square root is $a.bcd \cdots$ Continue until either the process terminates or the desired precision is reached.

Problem 4. Find the square root of 3 using Algorithm 1 up to 3 decimal places.

Problem 5. Find the square roots of

$$55225, 25281, 71824, 564752.25, 3972150625$$

using Algorithm 1. (These numbers are from the book *Nine Chapters*.)

Problem 6*. Use algebra to justify Algorithm 1, or tell a friend why the method works.

4.1.2 Algorithm 2: Shortening the Width

Assume $1 \leq x < 100$.

Step 1 Find the first digit of the square root, a. Let $b = x/a$.

Step 2 If $a = b$, terminate the process. If not, then $b > a$, go to Step 3.

Step 3 Calculate $\tilde{b} = b - (b-a)/2 = (a+b)/2$. Calculate $\tilde{a} = \frac{x}{\tilde{b}} = \frac{2x}{a+b}$.

Use $\tilde{a}$ and $\tilde{b}$ as new a, b and go to Step 2.

Repeat the process, the value a approximates the square root of x.

Problem 7. Use Algorithm 2 to find the square root of 3 accurate up to three decimal places.

Problem 8* Justify Algorithm 2 with algebra.

4.1.3 Algorithm 3: Continued Fraction

Assume that we want to find the square root of a number $x : 1 \leq x < 100$.

Step 1 Let a_1 be the first digit of the square root of x, $b = x - 1$.

Step 2 Calculate $a_2 = \frac{b}{2+a_1}$.

Step 3 Repeat Step 2: $a_n = \frac{b}{2+a_{n-1}}$

The sequence $\{a_1+1, a_2+1, a_3+1, \cdots, a_n+1, \cdots\}$ approaches $\sqrt{x}$. We may say, the *destination* of the sequence is $\sqrt{x}$.

Problem 9 Calculate

$$1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+1}}}}}}.$$

Problem 10* Explain why this continued fraction approaches the square root of $a+1$.

$$1+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cfrac{a}{2+\cdots}}}}}}.$$

4.1.4 Algorithm 4: Newton's Method, or the Ancient Egyptian Way

Assume that we want to find the square root of a number $x : 1 \le x < 100$.

Step 1 Let a_1 be the first digit of the square root of x.

Step 2 Calculate

$$a_2 = \frac{a_1}{2} + \frac{x}{2a_1}.$$

Step 3 Repeat Step 2

$$a_n = \frac{a_{n-1}}{2} + \frac{x}{2a_{n-1}}$$

until the desired precision is reached.

Problem 11. Use Algorithm 4 to find an approximate value of $\sqrt{3}$ with 3 decimal place precision.

Problem 12*. Explain why Newton's method works.

Problem 13*. Try to find a different algorithm for approximating the square root of a number between 1 and 100. Name the algorithm with your own name.

4.2 Algorithm to find the cubic root

Here is Newton's method of finding the cubic root of a number. The ancient Chinese way described in the book *Nine Chapters* is quite complicated. It is a generalization of Algorithm 1. You can perhaps try to rediscover it yourself.

4.2.1 Newton's method of finding the cubic root of x

We assume $1 \le x < 1000$.

Step 1 Let a_1 be the first digit of the cubic root of x.

Step 2 Calculate

$$a_2 = \frac{2a_1}{3} + \frac{x}{3a_1^2}.$$

Step 3 Repeat Step 2

$$a_n = \frac{2a_{n-1}}{3} + \frac{x}{3a_{n-1}^2}$$

until the desired precision is reached.

You perhaps can guess now that Newton's method should work in finding any root of a positive number x. Indeed, Newton's method can be used to find an approximate solution to any equation. It is one of the most useful method in modern numerical computation. Its general formula requires a simple use of a calculus concept, called the *derivative.*

Problem 14. Use Newton's method to find the first 4 significant digits of the cubic root of 3.

4.3 Finding any root with bisection method

Is there a way to find an approximate value of a solution to an equation such as

$$x^x = 3?$$

The answer is yes. But you may need to use your calculator a little bit.

Step 1 Estimate the solution: Find an interval $[a, b]$ so that $b^b > 3$ and $a^a < 3$.

Step 2 Try the midpoint of the interval now: calculate $(\frac{a+b}{2})^{(\frac{a+b}{2})}$. If the number is less than 3, call it a new a. If the number is greater than 3, call it a new b.

Step 3 Repeat Step 2 until the desired precision is reached.

Problem 15. Use the bisection method to find an approximate value of the solution to the equation

$$x^x = 3$$

accurate up to two decimal places.

Chapter 5

Engineering Consultation

The rule for a City Wall, Wall, Dyke, Trench, Moat and Canal is the same:

Add the upper and lower breadths, then halve; multiply by the altitude or the depth; then multiply by the length, giving the volume – Nine Chapters

In modern mathematics, finding the volume of a solid starts with the definition of the volume of a rectangular solid:

$$V = length \times width \times height.$$

Some simple solids' volumes can be obtained by using the method of trimming and patching. The rule given above can be justified easily with this method.

Problem 1 Now given a city wall with a lower breath of 12 meters and upper breath of 6 meters, an altitude of 15 meters and a length of 500 meters. Tell: what is the volume.

Unfortunately, most common solids' volumes can not be obtained with this trimming and patching method. These solids include the ball, the cone, the tetrahedron, the cylinder, etc.

Whether or not the volume of a regular tetrahedron can be obtained by using trimming and patching was one of the 23 problems Hilbert raised for the 20th century in 1900. The answer is no.

In this set of problems, we are allowed to use the volume formula for a pyramid of any shape: a cone, a tetrahedron, etc.

$$V = \frac{1}{3} height \times Base\ Area.$$

Problem 2 Find the volume of an icecream cone whose base perimeter is 24 cm and the height is 15 cm.

While it is easy to use the formula to find the volume of a cone, it is not so easy to find the volume of a frustum of a cone: a cone with the top part being cut off so that the circular top and the bottom are parallel.

Problem 3 Assume that we measure the top and bottom circumferences and we have 12 cm and 24 cm, respectively. The height is 10 cm. What is the volume of the frustum?

Problem 4* Find the formula for the volume of the frustum of a cone with the top and bottom circumferences C_1 cm and C_2 cm, respectively. The height is h cm.

We may think these two problems are easy. Let us try to challenge ourselves with some harder problems by cutting off the tip of the cone at an angle and then try to find the volume of the solid we have obtained. We may need the area formula of an ellipse: πab, where a, b are half of the long axis and the short axis, respectively.

Problem 5*** Assume that we have a cone with a base circumference C and a height h. Cut the tip of the cone off with at an angle so that the top is an ellipse. The highest point and the lowest point of the ellipse are h_1 and h_2. Find the volume of the rest of the cone.

Problem 6 Find the volume of a square pyramid whose base length is 100 meters and the height is 150 meters.

Problem 7 The tip of the pyramid in Problem 6 is again cut off. It is now a frustum with a height 120 meters. Find its volume.

Problem 8 Someone found a pyramid in a desert. But the top of the pyramid was gone. So it was really a frustum. He measured that the length of the bottom square was 100 meters and the length of the top square was

20 meters. The height of the frustum was 50 meters. What is the volume of this frustum?

Problem 9 A tetrahedron is a pyramid with a triangular base. It has total 4 triangular faces. Assume now we have a regular tetrahedron with the side length all equal to 10 cm. What is its volume?

Problem 10* We have a regular tetrahedron with the side length all equal to 10 cm. If we cut the top off at the 2/3 of the height. That is, the leftover frustum has a height equal to 2/3 of its original height. What is the volume of the frustum?

Problem 11** We have a beam that looks like prism – but two ends are trapezoids of different sizes. At one end, the upper width is 2 inches and the bottom width is 4 inches and the height is 4 inches. At the other end, the upper width is 3 inches and the bottom width is 6 inches and the height is 5 inches. Two trapezoids are parallel to each other and perpendicular to the beam in some sense. The length of the beam is 8 feet. Find its volume.

Problem 12* We now justify the volume formula of a cone. Assume the height is 9 and the radius of the bottom is 5.

We need to cut the cone horizontally into many thin disks. If we cut the cone into n disks – we have actually $n-1$ frustums and a tiny cone on the top. But anyway, we call all of them disks because we will treat them as if they were disks.

(1) What is the thickness (the height) of each disk?

(2) If we count from the top to the bottom, what are the radii of the first, the second, the third, and the fourth disk's bottoms, respectively?

(3) Treat each disk as if it were a cylinder with the radius of the base equal to the radius of the bottom of the disk. Find the approximate volumes of these disks.

(4) Add up the approximate volumes of these n disks and pull out the common factors including π. Do you see a sum of squares of whole numbers.

(5) Use the formula we had in Problem Set 2 to find the sum in (4).

(6) Find the destination of the number you get in (5) as n gets larger and larger – a calculator is needed now.

(7) The answer should be $75\pi = \frac{\pi r^2 h}{3}$.

Chapter 6

Even Transportation

While the titles for other chapters are more or less understandable, the title for this chapter is quite confusing and misleading. Ancient commentators explained that 'transportation' here means transportation of tax in the form of grain, such as millet. 'Even' here means that one needs to consider the distance one has to cover when transporting the tax to a tax collection center. However, this chapter discusses only very few problems remotely related to tax collection. Most problems are applications of fractions. These problems often appear in today's mathematics textbooks. We here select 15 out of 28 problems in this chapter. As we did in previous chapters, units of measurements are changed so readers can understand these problems better. But, since the numbers are kept the same, sometimes, such changes make these figures less realistic.

Problem 1 We are given that the task of collecting tax millet is distributed among 4 counties.

County A, 8 days away from the tax collection center, has 10,000 households;

County B, 10 days away from the center, has 9,500 households;

County C, 13 days away from the center, has 12,350 households;

County D, 20 days away from the center, has 12,200 households.

The total tax millet to be collected is 250,000 bushels. Assume that the tax collected should be *proportional* to the number of households in each county and *inversely proportional* to the distance from the tax center. How much each county should pay? If 10,000 carts are used to transport the tax millet and each cart can load 25 bushels. How many carts should be sent to each county to collect them? Find solutions in integers only.

Ans: County A, 83 100 bushels, 3324 carts; County B, 63 175 bushels, 2527 carts; County C, 63,175 bushels, 2527 carts; County D, 40 550 bushels, 1622 carts;

The next problem explains why households far away from the tax collecting center should pay less: the tax millet is, perhaps, transported to the center by farmers themselves. This was the case in the countryside where I was living in China during the 1970s. The tax rice or wheat were shipped to the tax collecting center by farmers using boats.

Problem 2. The tax collecting center is in County A, which has 20,520 households and where millet costs 20 dollars a bushel. Transportation cost is negligible.

County B, 200 miles away from the center, has 12,312 households and millet costs 10 dollars a bushel;

County C, 150 miles away from the center, has 7,182 households and millet costs 12 dollars a bushel;

County D, 250 miles away from the center, has 13,338 households and millet costs 17 dollars a bushel;

County E, 150 miles away from the center, has 5,130 households and millet costs 13 dollars a bushel.

The total tax to be collected is 10,000 bushels. A cart with capacity of 25 bushels costs 1 dollar per mile in transportation. Assume the payment by each household is the same in cash and labor. Find how much millet should each county pay.

Ans: $A : 3571\frac{517}{2873}; B : 2380\frac{2260}{2873}; C : 1388\frac{2276}{2873}; D : 1719\frac{1313}{2873}; E : 939\frac{2253}{2873}$.

Problem 3. Someone transports agricultural supplies from one place to another. An unloaded cart travels 70 miles a day and a loaded one travels 50 miles a day. This person makes 3 round trips in 5 days, how far is the distance between the two locations. (One needs to assume that one way is loaded and the other unloaded.)

Problem 4. A fast walker covers 100 yards, while a slow walker covers 60 yards. Assume that the latter goes 100 yards ahead of the former, who catches up with him. In how many yards will the two come abreast?

Problem 5. A slow walker goes ahead 10 miles. A fast walker, after pursuing 100 miles, is now ahead of the slow walker by 20 miles. In how many miles, does the fast walker catch up with the slow walker?

Problem 6. A hare runs 100 meters ahead. A hound, after pursuing 250 meters, is still 30 meters behind the hare. If the hound keeps pursuing, in how many more meters, will it catch up with the hare?

Problem 7. A person, carrying 12 pounds of gold through a pass, pays a tax of 10%. The pass takes away 2 pounds and gives back 5000 coins in return. Find, what is the price (in coins) of gold per pound?

Problem 8. A guest on horseback rides 300 miles a day. The guest left at day break and forgot his clothes. When the host discovered it, 1/3 day had just passed. The host started to chase the guest. As soon as he caught up with the guest, he gave back the clothes and turned back home. When he arrived home, 3/4 day had just passed. Assume neither the guest nor the host had stopped on the way. Find how far the host can go in a day.

Problem 9. Assume a cone-shaped lighthouse has 5 stories. The heights of 5 stories are in arithmetic progression. The bottom story is 40 feet and the top story is 20 feet. Find the heights of other stories.

Problem 10. 5 persons are to share 5 coins. The sum of the two greater shares is equal to that of the three smaller shares. Assume the shares are in arithmetic progression. How much does each get?

Problem 11. A wild duck flies from the south sea to the north sea in 7 days. A wild goose flies from the north sea to the south see in 9 days. Assume the two birds start at the same moment. When will they meet?

Problem 12. Peter starts from City A to City B, taking 5 days. Paul starts from City B to City A, taking 7 days. Assume Paul starts his journey 2 days earlier than Peter. When will they meet?

Problem 13. One person makes 38 prostrate tiles or 76 supine tiles a day. Assume he makes an equal number of both kinds of tiles in a day. How many tiles of each kind can he make?

Problem 14. A person can straighten 50 arrow shaft in one day, or pack feathers for 30 arrows, or install 15 arrow heads. Assume the he does all 3 jobs by himself. How many arrows can he prepare in a day?

Problem 15. A cistern is filled through 5 canals. Open the first canal and the cistern fills in 1/3 day. With the second canal open, it fills in one day. With the third, in 2 and a half days. With the fourth, in 3 days, and with the fifth, in 5 days. Assume all of the canals are opened. How many days are required to fill the cistern?

A cistern is a receptacle for holding water or other liquid, especially a tank for catching and storing rainwater.

Other Similar Problems

There are many similar problems in AMC8, AMC10, and AMC12. The next problem is sent by a friend of Sharon from Pennsylvania.

One day, Pauline was walking through a train tunnel on her way to town. Suddenly, she heard the whistle of a train approaching from behind her! Pauline knew that the train always traveled at an even 60 mile per hour. She also knew that she was exactly three-eighths of the way through the tunnel, and she could tell from the train whistle how far the train was from the tunnel. Pauline wasn't sure if she should run forward as fast as she could, or run back to the near end of the tunnel. Well, she did some lightening-fast calculations, based on how fast she could run and the length of the tunnel. She figured out that whichever way she ran, she would just barely make it out of the tunnel before the train reached her. How fast could she run? (Carefully explain how you found the answer.)

Chapter 7

Excess and Deficit

2000 years ago, people did not have the notation of algebra. They did not even have the negative numbers. But the problems they considered were very similar to the ones we have today. This chapter discusses how to solve certain type of systems of linear equations of two variables. But the concept of equations will not be introduced until next chapter. This chapter contains 20 problems. The first 8 problems explain how the rules work. The rest of the problems are exercises. The main method for solving these problems is the *Rule of Double False*. Again the context of some of the problems are changed. Try to use both algebraic and non-algebraic methods to solve these problems.

Problem 1. A group of people went to a restaurant. After the meal, they decided to divide the bill evenly among themselves. They found out if each paid 8 dollars, the excess was 3 dollars. If each paid 7, it was 4 dollars short (the deficit is 4). Tell: the number of people and the total cost of the meal.

Problem 2. Now chickens are purchased jointly by a group of people. If each contributes \$9, the excess is \$11. If each contributes \$6, the deficit is \$16. How many people are there and how much do the chickens cost?

Problem 3. A group of people go to an antique market to buy jade. If everyone contributes \$ $\frac{1}{2}$, the excess is \$4. If everyone contributes \$ $\frac{1}{3}$, the deficit is \$3. How many people are there in the group and how much does the jade cost?

Problem 4. Now cattle are purchased jointly. If every 7-household contribute 190, the deficit is $330. If every 9-household contributes 270, the excess is $30. How many households are there and how much do cattle cost?

Problem 5. Now Gold is purchased jointly. If everyone contributes $400, the excess is $3400. If If everyone contributes $300, the excess is $100. How many people are there and how much does the Gold cost?

Problem 6. Now sheep are purchased jointly. If everyone contributes $5, the deficit is $45. If If everyone contributes $7, the deficit is $3. How many people are there and how much do the sheep cost?

Problem 7. Now pigs are purchased jointly. If everyone contributes $100, the deficit is $100. If If everyone contributes $90, it is exactly the right amount. How many people are there and how much do the pigs cost?

Problem 8. Now dogs are purchased jointly. If everyone contributes $5, the deficit is $90. If If everyone contributes $50, it is exactly the right amount. How many people are there and how much do the dogs cost?

After these 8 problems, you may wonder it is possible to design a problem so that no solutions can be found? The answer is yes. We can come up with contradictory information so that solutions do not exist.

For example, let us change Problem 6 to the following:

Problem 6'. Now sheep are purchased jointly. If everyone contributes $5, it is $45 short of buying 3 sheep. If everyone contributes $10, it is $80 short of buying 6 sheep. Find the number of people in the group and the cost of each sheep.

We know immediately that it is impossible to have a solution since if everyone doubles the contribution to buy twice as many sheep, the deficit should be doubled too.

In algebra, this corresponds to a (non-homogenous) linear system that does not have a solution:

$$\begin{cases} ax + by = c \\ Ax + By = C. \end{cases}$$

If (A, B) is proportional to (a, b), then C must be proportional to c with the same ratio in order for a solution to exist.

But if we change \$80 to \$90, we will have many possible solutions (infinitely many, if fractions are allowed.)

The mathematics that deals with this type of problems is called *Linear Algebra*.

Try the following problem, which is not from the *Nine Chapters*. But it is a natural extension of these problems. Similar problems will be discussed in the next chapter.

Problem 9. Now chickens and pigs are purchased jointly. If everyone contributes \$10 to buy 10 chickens and 4 pigs, the deficit is \$10. If everyone contributes \$12, to buy 5 chickens and 5 pigs, the deficit is \$1. If every one contributes \$15, they can buy exactly 15 chickens and 5 pigs. Find out how many people in the group and how much each chicken and each pig cost.

What is interesting is that the Rule of Double False can be used to solve some problems we have already encountered.

Problem 10 . There is a wall of 9 feet. A gourd is planted above and its vine is creeping down 8 inches a day. A calabash is planted below and its vine is creeping up 1 foot a day. Find the number of days till they meet.

Method: Assume 5 days, the deficit is 8 inches short. Assume 6 days, the excess is 1 foot.

Problem 11. A wall is 5 meters thick. two rats tunnel from opposite sides. On the first day both the big and small rats tunnel a meter each. The big rat doubles its rate daily while the small rat halves its rate daily. Find the number of days till the two rats meet.

Problem 12. 5 large containers and 1 small container have a total capacity of 3 gallons. 1 large container and 5 small containers have a total capacity of 2 gallons. What is the capacity of each container?

Problems like these are numerous. From solving these problems, we can also appreciate the power of algebra.

Problem 13. There are certain number of chickens and rabbits in a cage. At the feeding time, one counts that there are total 35 heads. When one counts legs, there are 96. How many chickens and rabbits are there?

Some of you may have heard about the Chinese Remainder Theorem. Even though the problems sound similar, but they are solved very differently. The Chinese Remainder Theorem appeared in literature later than the *Nine Chapters*. Try to develop an algorithm to find the solutions of the next two problems.

Problem 14. Find the smallest positive integer x such that when it is divided by 8 the remainder is 6 and when it is divided by 9 the remainder is 7.

Problem 15. Find the smallest positive integer x such that when it is divided by 5 the remainder is 3 and when it is divided by 13 the remainder is 7.

Chapter 8

Rectangular Arrays

This is perhaps the most marvelous chapter among all nine chapters. I quote from the recent English translation and commentaries of the *Nine Chapters*: " This chapter makes a remarkable contribution to the development of mathematics." The Gaussian elimination method was clearly presented in this chapter in the general form. Gauss published his result in 1826, "about 2000 years later than the *Nine Chapters*". This chapter also introduces positive and negative numbers and rules of four basic operations.

The materials appeared in this chapter, solving systems of linear equations, will be studied in Algebra II and again in Linear Algebra when you go to college. However, in Algebra II, only systems of 3 unknowns are required. There are as many as 6 unknowns for some problems in this chapter.

The notation used in solving linear systems in this chapter coincides with the one we use today in Linear algebra: matrices. No variables are explicitly listed in the equations. Instead, the given data is arranged in the form of a *rectangular array* and operations are performed on the columns. We include a few problems in this set as an introduction to linear algebra. The Chinese language used to be written vertically. So, the given data is also displayed vertically, which is different from the convention in today's linear algebra textbooks.

Problem 1. Given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, and 1 bundle of low grade paddy, we gain 39 quarts of grain. 2

bundles of top grade paddy, 3 bundles of medium grade paddy, and 1 bundle of low grade paddy, yield 34 quarts of grain. 1 bundles of top grade paddy, 2 bundles of medium grade paddy, and 3 bundle of low grade paddy, yield 26 quarts of grain.

(Paddy: rice in the husk, growing, or gathered; rice in general; a rice field. In this problem, paddy means gathered ripe rice crop.)

Question: how much grain does one bundle of each grade paddy yield?

Solution: Top grade paddy yields $9\frac{1}{4}$ quarts per bundle. Medium grade paddy yields $4\frac{1}{4}$ quarts per bundle. Low grade paddy yields $2\frac{3}{4}$ quarts per bundle.

Method: List the given data as a rectangular array:

$$\begin{matrix} TopGrade \\ MediumGrade \\ LowGrade \\ quarts \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{bmatrix}$$

Perform the following operations on the columns to reduce the rectangular array to the desired form: the left upper triangle has only zero entries.

1. Middle $\times$ 3 - Right $\times$ 2. We have

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{bmatrix}$$

2. Left $\times$ 3 - Right.

$$\begin{bmatrix} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{bmatrix}$$

3. Left $\times$ 5 - Middle $\times$ 4.

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{bmatrix}$$

From the last rectangular array, we can easily get the answer for the low grade rice crop first. Then, use successive substitution to get answers for the medium and top grade rice crop.

In order for this method to work for general problems, it is necessary now to introduce the negative numbers and their operations. Ancient Chinese used counting rods of different colors to stand for positive (red) and negative (black) numbers. The color code is just the opposite of what we use today. Since I am not using colors in this document, I will use the negative sign for negative numbers.

Problem 2. Replace the data in Problem 1 with the one given below and find the answers to the same question.

$$\begin{matrix} TopGrade \\ MediumGrade \\ LowGrade \\ quarts \end{matrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now, try a problem with negative entries.

Problem 3. If we sell 2 cattle and 5 sheep to buy 13 pigs, we have a surplus 1000 coins. If we sell 3 cattle and 3 pigs, we can buy exactly 9 sheep. If we sell 6 sheep and 8 pigs, we still need 600 coins to buy 5 cattle. Find the price of each animal.

The data gives the following rectangular array.

$$\begin{matrix} Cattle \\ Sheep \\ Pig \\ coins \end{matrix} \begin{bmatrix} -5 & 3 & 2 \\ 6 & -9 & 5 \\ 8 & 3 & -13 \\ -600 & 0 & 1000 \end{bmatrix}$$

Problem 4. If you think these problems are fun to solve, here is another one. But I have removed the context of the problem.

$$\begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & -1 \\ 4 & -1 & 0 \\ 1 & 1 & 11 \end{bmatrix}$$

If you like to set up the rectangular array yourself, try the following problem.

Problem 5. 1 county magistrate, 5 officials, and 10 servants can eat 10 chickens in total. 10 magistrates, 1 official, and 5 servants can eat 8 chickens. 5 magistrates, 10 officials, and 1 servant can eat 6 chickens.

Question: how many chickens can each of them eat?

There are also indeterminate systems in this chapter: the number of unknowns is bigger than the number of equations. There are also total 18 problems.

Chapter 9

Right-Angled Triangles

This chapter talks about the proofs and applications of the so-called "Pythagoras" theorem. It is called "Gougu" theorem in Chinese (even in today's textbooks). We talked about this theorem in the very first chapter. "Gou" and "Gu" refer to the two sides of a right-angled triangle.

Let c be the hypotenuse and a, b be the two sides. There are a couple of interesting identities that are equivalent to the "Gougu" theorem ($c^2 = a^2 + b^2$). Try to use your algebra skills to show that they are true identities.

$$2(c+a)(c+b) = (a+b+c)^2.$$

$$2(c+a)(c-b) = (a-b+c)^2.$$

For our high schoolers, try to solve the following problems.

Let c be the hypotenuse and a, b be the two sides.

1. Show that a must be a root of the cubic equation

$$x^3 + \frac{c-a}{2}x^2 = \frac{a^2b^2}{2(c-a)}.$$

2. Show that b must be a root of the biquadratic equation

$$x^4 + a^2x^2 = c^2b^2$$

Problem 1 Given a large circular log with a diameter 2.5 meters, assume it is turned into a rectangular plank of 0.7 meters thick. What is the (maximal) width?

Problem 2 Given a tree 20 meters high and 1 meter in circumference, a vine winds around it (evenly) 7 times from its root to its top. Find the length of the vine.

Problem 3 There is a reed in a pond. Its top is 1 meter above the water and is 5 meters from the bank. When it is drawn to the bank, its tip can barely touch the bank. Find the depth of the water and the length of the reed.

Problem 4 There is a rope hanging from the top of a pole. The extra length lying on the ground is 3 meters long. When tightly stretched, it is 8 meters from the foot of the pole. Find the length of the rope.

Problem 5 There is a wall 10 meters high. A pole leans against the wall so that its top is even with the top of the wall. If the foot of the pole is moved 1 meter further from the wall, the pole will lie flat on the ground. Find the length of the pole.

Problem 6 There is a circular log of unknown size (buried in a wall). When sawn (along the length) 0.1 meter deep, it shows a breadth of 1 meter. Find the diameter of the log.

Problem 7 There is a gate. When partially opened, it is 1 meter away from the threshold and the the gap between the halves is 0.2 meter. Find the width of the gate.

Problem 8 There is a door whose diagonal is 4 feet longer than the width and 2 feet longer than its height. Find the size of the door.

Problem 9 There was a bamboo 10 meters high. It was broken during a storm and now its tip is on the ground 3 meters away from its root (it is connected at the breaking point). Find the height of the breaking point.

Problem 10 There is a right-angled triangle with two sides 5 and 12. Find the side of the inscribed square.

Problem 11 There is a right-angled triangle with two sides 8 and 15. Find the diameter of the inscribed circle.

Problem 12** Find the side of the inscribed square of a general triangle with sides a,b,c.

Problem 13** Find the diameter of the inscribed circle of a general triangle with sides a,b,c.

Problem 14 A castle is surrounded by tall walls forming a square of sides 200 meters each. Four gates are located in the middle of each wall. At 15 meters outside the east gate, there is a tree. How far do people need to walk south from the south gate in order to see the tree (assuming there is no obstacles other than the wall.)?

Problem 15 There is a square city of unknown side, with gates opening in the middle of each side. 20 meters north of the north gate there is a tree, which is visible when one goes 14 meters outside the south gate and then 1775 meters westward. Find the length of each side (of the city).

Problem 16 (Paraphrased from Sharon's piano teacher Anne Listokin) There is a deep moat around a square city. The moat is 15 feet wide. A person has only two 14 feet long planks of wood (strong enough for him to walk on) and nothing else. Can he walk cross the moat using these two planks? (assume the land outside the city is flat, and the four corners of the moat are right-angled.)