Eric's Comments on Problem Set 4.

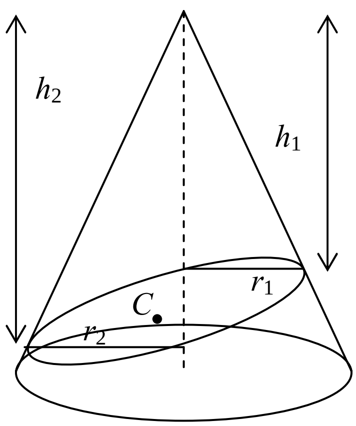
Problem 3 & 4:

These problems are so similar, it is easier to do problem 4 and then use it to solve problem 3. Unfortunately, the problems are worded in terms of the circumference, instead of the radius or the area. If you use the area, the formula comes out a bit nicer, in my opinion:

$$V = \tfrac{1}{3}\left(A_1 + \sqrt{A_1 A_2} + A_2\right)h$$

The nice thing about this formula is that it applies (and can be proven) for *arbitrary* bases, not just bases that are circles. So you can get interesting frustums of all kinds of shapes. Also, this formula can help us find the volume of an arbitrary cone (see problem 12 below). In terms of radius, the formula is

$$V = \tfrac{1}{3}\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)h .$$



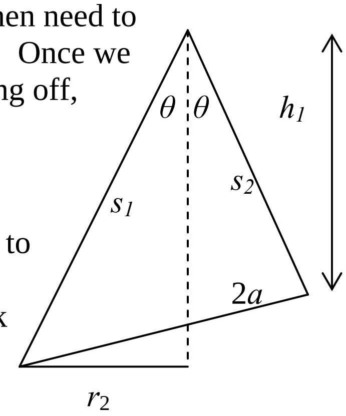
Problem 5:

This one is quite difficult! I used a trick that helped me a lot to simplify this, but it was still really nasty. Although Dr. Jiang says that finding the semi-minor axis is the hard part, I found this the easiest part.

First, it is easier to work with the heights h_1 and h_2 as I've illustrated them in the figure at right. Then it isn't much work to find formulas for the two distances labeled r_1 and r_2. This allows us to find the position of the center C of the tilted elipse illustrated at right. It lies a distance $x = \tfrac{1}{2}\left(r_2 - r_1\right)$ to the left of the dashed line, and the radius of the cone at this height is $r = \tfrac{1}{2}\left(r_1 + r_2\right)$. It then isn't too hard to figure out the distance b that one must move from this midpoint to intersect the wall of the cone, since the distance from the dashed line is just $r = \sqrt{x^2 + b^2}$.

Now that we have the semi-minor axis b of the ellipse, we then need to find the semi-major axis a and then the height of this elliptical base. Once we put all this together, we can find the volume of the part we are cutting off, which is just

$$V = \tfrac{1}{3}\pi abh' ,$$



where *h'* is the height of the cut off part, as measured perpendicular to the elliptical base. Now, look at the triangle sketched at right. The sloped base is $2a$, and the height of this triangle is *h'*, so if you think about it, *ah'* is just the area of this triangle, so we can rewrite the volume as

$$V = \tfrac{1}{3}\pi Ab$$

where A is the area of the triangle. Then, with a little knowledge of trigonometry, the area of this triangle can also be written as

$$A = \tfrac{1}{2} s_1 s_2 \sin(2\theta),$$

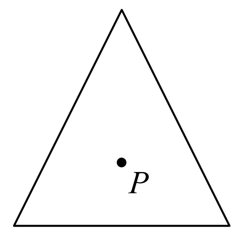
where 2θ is the apex angle of the triangle. A little more trigonometry and algebra, and soon you'll have the formula for the cut off portion of the cone, namely

$$V = \pi r^2 (h_1 h_2)^{3/2} \big/ 3h^2$$

You can then rewrite this in terms of the original definitions of h_1 and h_2, if you want to, and then subtract it from the volume of the total cone to get the portion you want.

Problem 9 & 10:

Problem 9 requires a bit of geometry to find the height of a regular tetrahedron. If you call the edge length s, then it isn't too hard to find the area of an equilateral triangle of side s. If you then realize that the point P is one-third of the way up from the base to the apex of the triangle, and that the last vertex of the tetrahedron is directly above P, you should be able to use the distance formula to figure out how high the tetrahedron is. Write your answer in terms of s in general, before plugging in specific numbers.



Problem 10 is very easy once you have problem 9. Simply notice that the piece you are removing is a regular tetrahedron one-third the size of the original.

Problem 11

As I have communicated to Dr. Jiang, this problem is ill-defined. The shapes of intermediate cross-sections is not defined. It is not even clear if the shape described is a polyhedron at all. It is impossible to get a "correct" answer to an inadequately defined problem, so I recommend not doing this problem.

Problem 12

The solution that Dr. Jiang is recommending for this problem is nice, and not too hard (though a calculator is not really needed for part 6). The method he is recommending is basically using a calculus concept called integration. It is straightforward, and it works.

Here is another interesting method. Consider a cone with arbitrary base of area A_1 and height h_1. The base need not be a circle. It should be obvious that if you stretch the cone by increasing h_1, then the volume will increase proportionally. Similarly, if you stretch the base by, say, increasing both dimensions by a factor of 2, then both the area and the volume will increase by a factor of 4. In other words, the volume of this cone will be

$$V_1 = kA_1 h_1$$

where the constant k is unknown. Now, consider taking this cylinder and cutting off a portion of the top to form a frustum. The piece you cut off will be given by

$$V_2 = kA_2 h_2$$

The final volume will be the difference between these two,

$$V = k\left(A_1h_1 - A_2h_2\right)$$

Now, the big cone you started with and the little cone you cut off are clearly similar. This means that the area of their bases will be proportional to the square of their heights. In other words,

$$A_1/A_2 = h_1^2/h_2^2 \ .$$

With some work, you can do some algebra to then rewrite the volume of the frustum purely in terms of the two areas and the height of the frustum, to yield the formula

$$V = k\left(A_1 + \sqrt{A_1A_2} + A_2\right)h$$

This is just the formula from questions 3 and 4, but with k unknown.

Now, I started this discussion by claiming that I was going to find the formula for the volume of a cone. I now have the volume for a frustum of a cone *if* I knew the constant k. But there is *one* circumstance where I can find k. Suppose the top and bottom have the same area, so that our frustum is really a cylinder. Then using our formula, we have

$$V = 3kAh$$

But the correct formula for a cylinder is just $V = Ah$, so we must have $3k = 1$, or $k = 1/3$. And thus, by considering frustums, we found the volume of a cone!

Volume of a Sphere

The volume of a sphere did not appear in this problem set, but it is surprisingly easy. Consider the three objects below: a sphere of radius r; a double-cone, each cone with radius r and height r; and a cylinder of radius r and height $2r$. Consider intersecting these three objects with a common plane, as illustrated below. It is not too difficult to show that the sum of the area of the first two circles is equal to the area of the third circle. Since volume is just sort of the sum of the area of each "slice" through the objects, it follows that the sum of the volume of the sphere and the double cone equals the volume of the cylinder, and we can find the volume of a sphere in a snap.