

Game One Solution:

1. If  $n = 2, 4, 8, \dots, 2^m$ , Player B has a winning strategy.

Since Player A can not take all. Player A may take  $A_1 = 2^k p$ , where  $k < m$  and  $p$  is an odd number. So, Player B has  $2^m - 2^k p = 2^k(2^{m-k} - p) = 2^k q$ , where  $q$  is an odd number. If  $q = 1$ , Player B can take  $2^k \leq A_1$  and wins. If  $q > 1$ , then Player B is in the situation of Player A when the starting number is not a power of 2. Player B wins by following the strategy described below.

2. If  $n$  is not a power of 2, Player A has a winning strategy.

Player A starts with  $2^m p$ , where  $p \geq 3$  is an odd number. Player A takes  $2^m$ . Player B has  $2^m(p - 1)$  sticks left. Since  $p - 1 \geq 2$  and Player B can not take more than  $2^m$ , so Player B can not take all and will not win in this round.

Assume now Player B takes  $2^k q$  sticks, where  $k < m$  and  $q$  an odd number. Player A has  $2^m(p - 1) - 2^k q = 2^k(2^{m-k}(p - 1) - q)$  sticks left. If  $2^{m-k}(p - 1) - q = 1$ , then Player A wins by taking all  $2^k$  sticks. If  $2^{m-k}(p - 1) - q > 1$ , since it is an odd number, we go back to the starting point when Player A had  $2^m p$  sticks, where  $p \geq 3$  is an odd number. Repeating what Player A did in the first round, until Player A takes all and wins.

Note: one might think that Play A's winning strategy is to bring the number down to the nearest power of 2. For example, if  $n = 14$ , then, A takes 6 to bring the number down to 8. Player A will win this game. But such strategy does not work for large numbers. For example, if  $n = 36$ , A takes 4 to leave 32 to B. If B takes 4, leaving A with 28. Player A will have to take 4 to win. Player A can not take 12 to bring the number down to 16 since it will violate the rule. The move to take the number down to the nearest power of two works only when such move does not violate the rule - for example, in the first round.

Game two (Fibonacci Nim) solution:

1. If  $n = 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$ , the Fibonacci numbers, Player B has a winning strategy.

If  $n$  is a Fibonacci number, Player A can not take all the sticks. If Player A takes a number of sticks and leaves Player B with another Fibonacci number  $m$  of sticks, then,  $m \leq 2(n - m)$ . So, Player B wins. If Player A leaves a non-Fibonacci number of sticks to Player B. Player B can follow the strategy described in the next section to win.

2. If  $n$  is a non-Fibonacci number, Player A has a winning strategy.

For any non-Fibonacci number  $n$ , we let

$$F_1 = \max\{F : F < n, F \text{ is Fibonacci}\}.$$

$$F_2 = \max\{F : F \leq n - F_1, F \text{ is Fibonacci}\}.$$

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$$F_k = \max\{F : F \leq n - F_{k-1}, F \text{ is Fibonacci}\}.$$

We have Fibonacci numbers  $F_1, F_2, F_3, \dots, F_k$  and  $n = F_1 + F_2 + \dots + F_k$ ,  $k \geq 2$ .

Since  $n$  is finite, this sequence ends in finite steps. For example,

$$70 = 55 + 15 = 55 + 13 + 2.$$

$$101 = 89 + 12 = 89 + 8 + 4 = 89 + 8 + 3 + 1.$$

Winning strategy for Player A:

To start the game, take  $F_k$  of sticks. For subsequent turns, take all the sticks to win or repeat the strategy described above.